A Sampling Approach to Extreme Value Distribution for Time-Dependent Reliability Analysis

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Maintaining high accuracy and efficiency is a challenging issue in time-dependent reliability analysis. In this work, an accurate and efficient method is proposed for limit-state functions with the following features: The limit-state function is implicit with respect to time. There is only one stochastic process in the input to the limit-sate function. The stochastic process could be either a general strength or a general stress variable so that the limit-state function is monotonic to the stochastic process. The new method employs a sampling approach to estimate the distributions of the extreme value of the stochastic process. The extreme value is then used to replace the corresponding stochastic process. Consequently the time-dependent reliability analysis is converted into its time-invariant counterpart. The commonly used time-invariant reliability method, the first order reliability method, is then applied to calculate the probability of failure over a given period of time. The results show that the proposed method significantly improves the accuracy and efficiency of time-dependent reliability analysis. [DOI: 10.1115/1.4023925]

Keywords: time-dependent reliability, stochastic process, extreme value, first order reliability method

1 Introduction

Time-dependent reliability analysis quantifies the probability that a component or system survives after it has worked for a certainty time t or over the period $[0, t]$ [1]. Time-dependent reliability in general decreases with time for two reasons. First, there exist time-dependent uncertainties. For example, the wave loading on offshore structures always changes its amplitude and direction over time [2]. For this kind of time-dependent uncertainty, its global extreme over time varies with the length of the time period. More specifically, a longer time period will more likely correspond to a larger global extreme. On the other hand, the strengths of a component or system generally deteriorate over time. The other reason is that a limit-state function, which predicts the state of safety or failure, may also be a function of time. For example, the motion error of a mechanism varies at different instants of motion input [3]. Because of time-dependent uncertainties, a product always exhibits time-dependent failures, and the associated reliability is almost always declines over time.

Limit-state functions for the time-dependent reliability analysis can be classified into the following three categories:

- Limit-state functions have time-invariant random variables $\mathbf{X}$ and time t in its input. A function in this category is therefore in the form of $G = g(\mathbf{X}, t)$, where $g(\cdot)$ is the limit-state function, and G is a response variable.
- Limit-state functions are in the form of $G = g(\mathbf{X}, \mathbf{Y}(t))$ with time-invariant random variables $\mathbf{X}$ and time-dependent stochastic processes $\mathbf{Y}(t)$, but are implicit with respect to time t.
- Limit-state functions are explicit with respect to time t and are functions of $\mathbf{X}$ and $\mathbf{Y}(t)$. This is the most general case where $G = g(\mathbf{X}, \mathbf{Y}(t), t)$.

In the past decades, many progresses have been made in time-dependent reliability analysis methodologies. They include the Monte Carlo simulation (MCS) [4], the Gamma process method [5], the Markov chain method [6], the extreme value distribution method [7,8], the upcrossing rate method [9,10], and the composite limit state method [1]. Among them, the most popular method is the upcrossing method. This method is based on the Rice's formula [11], whose key step is the computation of the upcrossing rate. This method has advantages over other methods for its efficiency. But its accuracy is poor for problems whose reliability is low. The accuracy issue comes from the assumption that all the upcrossings are independent. Many improvements on the Rice's formula have been made to improve its accuracy. The recent developments include the work of Sudret [12], where an analytical upcrossing rate is derived. This upcrossing rate is further derived by Zhang and Du [13] later for kinematic reliability analysis of function generator mechanisms. Mejri and Cazuguel [14] also combine the upcrossing rate method with the finite element analysis calculations to assess the time-variant reliability of marine structures.

In spite of many progresses, maintaining high accuracy and efficiency is still a challenge. The MCS method can evaluate the time-dependent reliability accurately, but the required computational cost may not be affordable. As mentioned previously, the widely used upcrossing rate method is relatively efficient, but its accuracy may be poor [15]. It assumes that all the upcrossings are statistically independent. The assumption is conservative and may produce large errors. Several empirical modifications have been made by Vanmarcke [16] and Preumont [17] to remedy the drawbacks of the independent upcrossing assumption. These modifications, however, are limited to special problems.

For special problems where there is only one loading stochastic process in the input, the extreme value of the stochastic process can be used to replace the process. This work is concerned with a sampling approach to the extreme value of a stochastic process $Y(t)$, which can be a load or a strength. The basic idea is to replace $Y(t)$ in the limit-sate function with its extreme value and then convert the time-variant problem into its time-invariant counterpart. The samples of the extreme value of $Y(t)$ are obtained from MCS.

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Contributed by the Design Automation Committee of ASME for publication in the Journal of Mechanical Design. Manuscript received December 30, 2012; final manuscript received February 8, 2013; published online May 10, 2013. Assoc. Editor: David Gorsich.

It is then used to estimate the distribution of the extreme value. Saddlepoint approximations are employed [18] for the distribution estimation.

The new method is developed for special limit-state functions in category two. An eligible limit-state function for the new method is in the form of $G = g(\mathbf{X}, Y(t))$; and the stochastic process $Y(t)$ is either a general strength variable, meaning that $g(\cdot)$ decreases when the process increases, or a general stress variable, meaning that $g(\cdot)$ increases when the process increases. This special type of limit-state functions appears in many important applications. For example, the limit-state functions associated with the following response variables fall into the special type: the stress of hydrokinetic turbine blades under random wave loading [19], the damage of bridge under random traffic loading [20], and the deflection of beam under loading with stochastic disturbance [21].

Once the extreme value of $Y(t)$ over $[0, t]$ is available, it is used to replace $Y(t)$. Since the extreme value is a random variable, the limit-state function becomes time invariant. Any time-invariant reliability methods can then be applied for the reliability evaluation. In this paper, we use First Order Reliability Method (FORM) as an example for the time-invariant reliability analysis since it is the most widely used reliability analysis method. For those limit-state functions, which are very nonlinear and have multiple most probable points (MPP), other time-invariant reliability methods can be employed to substitute FORM.

The major contributions of this work are multifold: (1) With the use of the extreme value of a general strength or general stress variable, the independent upcrossing assumption of the Rice's formula is completely removed, and then the accuracy of time-dependent reliability analysis is also improved significantly. (2) We employ the saddlepoint approximation in estimating the distribution of the extreme value of the general strength or general stress processes. The employment makes the sampling procedure not only accurate but also robust. (3) Given the good accuracy and efficiency, the use of the proposed method will also make time-dependent reliability-based design more accurate and efficient.

In Sec. 2, we review time-dependent reliability analysis. We then introduce the new method in Sec. 3. In Sec. 4, we provide two examples, then followed by conclusions in Sec. 5.

2 Methodology Review

In this section, we review the definition of time-dependent reliability. We also discuss the upcrossing rate method. This method is most commonly used and will be compared with the proposed method in the two examples.

2.1 Time-Dependent Reliability. Reliability is the probability that a component or system performs its intended function under intended conditions over a certainty period of time. A limit-state function $G = g(\mathbf{X}, \mathbf{Y}(t))$ defines the intended conditions; it also determines if the intended function is performed properly as follows: if $g(\cdot) < 0$, the intended function is performed properly; if $g(\cdot) > 0$, the intended function is not performed properly, and then a failure occurs. Let the time period be $[0, t_s]$. The time-dependent reliability is given by

$$R(0, t_s) = \Pr\left\{g(\mathbf{X}, \mathbf{Y}(t)) < 0, \quad \forall t \in [0, t_s]\right\} \tag{1}$$

The associated time-dependent probability of failure is

$$p_f(0, t_s) = \Pr\left\{g(\mathbf{X}, \mathbf{Y}(t) > 0, \quad \exists t \in [0, t_s]\right\} \tag{2}$$

It should be note that $p_f(0, t_s)$ in Eq. (2) is defined for a given time period $[0, t_s]$. It will change as the time period changes.

It is well-known in reliability engineering that the time-dependent reliability can be derived from the failure rate by [22]

$$p_f(0, t_s) = 1 - \exp\left\{-\int_0^{t_s} \lambda(t)dt\right\} \tag{3}$$

where $\lambda(t)$ is the failure rate at time instant t. $\lambda(t)$ is defined by the following limit of a conditional probability

$$\lambda(t) = \lim_{\Delta t \to 0} \Pr\{t < T < t + \Delta t)|T > t\}/\Delta t \tag{4}$$

where T is the time to failure, which is apparently a random variable. Given the limit-state function, we also have

$$\lambda(t) = \lim_{\Delta t \to 0} \Pr\left\{g(t + \Delta t) > 0 \middle| g(t) < 0, \quad t \in [0, t_s]\right\}/\Delta t \tag{5}$$

where the notation $g(t)$ is used for $g(\mathbf{X}, \mathbf{Y}(t))$.

The conditional probability in Eqs. (4) or (5) is the probability of failure at t given that no failures have occurred prior to t.

2.2 Upcrossing Rate Method. Equation (3) indicates that $p_f(0, t_s)$ is obtainable by integrating the failure rate $\lambda(t)$ over $[0, t_s]$. The upcrossing method uses the upcrossing rate $v^+(t)$ to approximate the failure rate $\lambda(t)$; namely

$$\lambda(t) \approx v^+(t) \tag{6}$$

an upcrossing is defined as an event when the limit-state function crosses the limit state from the safe region $g(\cdot) < 0$ to the failure region $g(\cdot) > 0$.

We now briefly review how to compute the upcrossing rate using FORM, and more details can be found in Refs. [13,23].

With FORM, the limit-state function is transformed as follows:

$$G = g(\mathbf{X}, \mathbf{Y}(t)) = g(T(\mathbf{U}_X), \quad T(\mathbf{U}_Y(t))) = h(\mathbf{U}_X, \mathbf{U}_Y(t)) \tag{7}$$

where $T(\cdot)$ stands for the transformation.

Then FORM searches for the most probable point (MPP). The limit-state function is then linearized at the MPP, $\mathbf{U}^*(t) = (\mathbf{U}_X^*, \mathbf{U}_Y^*(t))$, which is the shortest distant point between $h(\cdot) = 0$ and the origin. The upcrossing rate $v^+(t)$ is then computed by Ref. [11]

$$v^+(t) = \omega(t)\phi(\beta(t))\{\phi[\dot{\beta}(t)/\omega(t)] - [\dot{\beta}(t)/\omega(t)]\Phi[-\dot{\beta}(t)/\omega(t)]\} \tag{8}$$

in which

- $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function (PDF) and cumulative distribution function (CDF) of a standard normal variable, respectively.
- $\omega(t)$, $\beta(t)$, and $\dot{\beta}(t)$ are variables computed using $\mathbf{U}^*(t)$ and the correlation function $\rho(t_1, t_2)$ of $\mathbf{Y}(t)$ [23].

Once $v^+(t)$ is obtained, we can use Eq. (3) to calculate $p_f(0, t_s)$. However, the equation is based on the assumption that the initial probability of failure at $t = 0$ is zero. After accounting for the initial probability of failure, $p_f(0)$, we have the following equation [13]

$$p_f(0, t_s) = 1 - [1 - p_f(0)]\exp\left\{-\int_0^{t_s} v^+(t)dt\right\} \tag{9}$$

The upcrossing method has two drawbacks. Its accuracy may not be satisfactory because of the independent upcrossing assumption, which may not hold for many applications, especially when the time interval is long and when there are many upcrossings. The other drawback is the demanding computation. FORM must be performed at all the time instants required by the integral in Eq. (9). In the next section, we develop a new method to overcome the two drawbacks.

3 The Sampling Methodology

The central idea is to convert the time-dependent problem into a time-invariant problem for limit-state functions with only one stochastic process. Then the time-invariant reliability method, such as the FORM, can be used. We first give the general principle of the new sampling method and then provide its detailed steps.

3.1 Overview. To use the time-invariant reliability method, we make the following assumptions:

- The limit-state function is implicit of time. In other words, time t does not explicitly appear in $g(\cdot)$.
- The stochastic process $Y(t)$ is either a general strength variable or a general stress variable.

A general strength variable is a variable related to resistance to loading. For example, the yield strength of materials is a strength variable. When the general strength variable increases, $g(\cdot)$ decreases. A general stress variable is a variable related to load. For example, the external force or moment is a stress variable. When the general stress variable increases, so does $g(\cdot)$.

The above assumption is reasonable for a wide range of engineering applications. For example, in many dynamic systems, time t does not appear explicitly in the dynamics equations, but the equations are still time-dependent because they involve time-dependent coefficients $Y(t)$. For many engineering applications, a large number of stochastic processes may be rarely encountered. Obtaining the complete information about a stochastic process is costly and time consuming. In many cases, only the most important stochastic process, such as a load, is considered. Then resources are allocated for collecting necessary data for the selected process. With only one stochastic process, it is easy to identify whether it is a general strength or a general stress variable before the reliability analysis is performed.

With the assumption, a limit-state function is given by

$$G = g(\mathbf{X}, Y(t)) \tag{10}$$

The probability of failure can then be evaluated by

$$\begin{aligned} p_f(0, t_s) &= \Pr\{g(\mathbf{X}, \mathrm{Y}(t) > 0, \quad \exists t \in [0, t_s]\} \\ &= \Pr\{\max g(\mathbf{X}, \mathrm{Y}(t)) > 0, \quad \forall t \in [0, t_s]\} \\ &= \Pr\{g(\mathbf{X}, W) > 0\} \end{aligned} \tag{11}$$

where W is the global extreme value of $Y(t)$ over [0, t_s], It is given by

$$W = \min\{Y(t),\ t \in [0, t_s]\} \tag{12}$$

for a general strength variable, and

$$W = \max\{Y(t),\ t \in [0, t_s]\} \tag{13}$$

for a general stress variable.

We have now converted the time-variant problem into a time-invariant one as indicated in Eq. (11). Then, a time-invariant reliability method can be applied.

3.2 Procedure. Two stages are involved in the new method. The first is to obtain the distribution of the extreme value of the stochastic process. The second is the time-invariant reliability analysis. The procedure is summarized in Fig. 1, and the details are given below.

Stage 1: Estimate the distributions of extreme values of $Y(t)$ by MCS.

Since we do not need to call the limit-sate function in this stage, we can simply use MCS. Denote the CDF of W by $F_W(w)$, which can be obtained with the following three steps:

Step 1: Divide the time interval [0, t_s] into discrete points $(t_0, t_1, \cdots, t_N = t_s)$ or N equal small intervals. $Y(t_i)$ $(i = 0, 1, \ldots, N)$ is a random variable at t_i. Generate n random samples for $Y(t_i)$. The samples or trajectories of $Y(t)$ are denoted by $Y_j = (y_j(t_0), \cdots y_j(t_i) \cdots y_j(t_N))$ $(j = 1, 2, \cdots, n)$, or $Y_j = (y_j(t_i))_{i=0,1,\ldots,N}$, where n is the number of simulations.

Step 2: Identify the extreme values of the trajectories of $Y(t)$. If $Y(t)$ is a general strength variable, the minimum value is found by $w_j = \min_i (y_j(t_i))_{i=0,1,\ldots,N}$. If $Y(t)$ is a general stress variable, the maximum value is found by $w_j = \max_i (y_j(t_i))_{i=0,1,\ldots,N}$.

Step 3: estimate the CDF F_W (w) from the samples w_j $(j = 1, 2, \ldots, n)$.

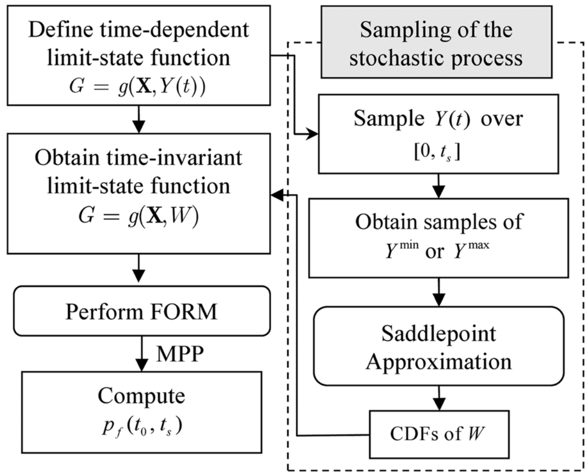

Fig. 1 Flow chart of the new time-dependent reliability method

Once the distributions of W are available, we replace $Y(t)$ with W, and the limit-state function becomes

$$G = g(\mathbf{X}, W) \tag{14}$$

The details of this stage will be discussed in Secs. 3.3 and 3.4

Stage 2: time-invariant reliability analysis

In this work, we use FORM. Other reliability methods can also be used. The use of FORM will be presented in Sec. 3.5.

3.3 Sampling Approach to the Extreme Values of Stochastic Processes. Our task now is to obtain samples of $Y(t)$ and use them to estimate the CDF of the extreme value $F_W(w)$. We take a Gaussian process as an example. For this process, we use the expansion optimal linear estimation method (EOLE) [24] to generate samples. The approach approximates $Y(t)$ with a finite set of random variables. Let the mean function and standard deviation function of $Y(t)$ be $\mu_Y(t)$ and $\sigma_Y(t)$, respectively, and assume that the autocorrelation coefficient function is $\rho_Y(t_1, t_2)$. At first, the time interval [0, t_s] is divided into s time points $(t_i)_{i=1,2,\cdots,s} = (t_1 = t_0, t_2, \ldots, t_s)$. After the discretization, the correlation matrix of the time points $(t_i)_{i=1,2,\cdots,s}$ is computed by

$$\sum = \begin{pmatrix} \rho_Y(t_1, t_1) & \rho_Y(t_1, t_2) & \cdots & \rho_Y(t_1, t_S) \\ \rho_Y(t_2, t_1) & \rho_Y(t_2, t_2) & \cdots & \rho_Y(t_2, t_S) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_Y(t_S, t_1) & \rho_Y(t_S, t_2) & \cdots & \rho_Y(t_S, t_S) \end{pmatrix}_{s \times s} \tag{15}$$

Let, η_i and $\boldsymbol{\varphi}_i^T$ be the eigenvalues and eigenvectors of the matrix, respectively. Then the expansion of the stochastic process is given by [24]

$$Y(t) = \mu_Y(t) + \sigma_Y(t) \sum_{i=1}^{p} \frac{U_i}{\sqrt{\eta_i}} \boldsymbol{\varphi}_i^T \boldsymbol{\rho}_Y(t) \tag{16}$$

in which $\boldsymbol{\rho}_Y(t)$ is a vector of correlation coefficients with j-th element be $\rho_Y(t, t_j)$, $j = 1, 2, \cdots, p \le s$, U_i $(i = 1, 2, \cdots, p \le s)$ are independent standard normal random variables. We can then generate random samples of U_i and use them to reproduce sample curves (trajectories) of $Y(t)$ using Eq. (16). More details about the EOLE can be found in Ref. [24].

After n simulations, we obtain n trajectories of the stochastic process sample. Figure 2 shows such a trajectory, from which we can easily find the extreme values.

With the obtained extreme value samples w_j $(j = 1, 2, \ldots, n)$, we now discuss how to estimate the CDF $F_W(w)$, which is needed in FORM for the random variable transformation. The tail areas $F_W(w)$ are usually involved, and this requires the estimation of small probabilities and a large of number of simulations.

The most straightforward way is to use the empirical CDF by sorting w_j such that $w_{(1)} < w_{(2)} < \ldots < w_{(n)}$. Then we can use the following equation:

$$F_W(w_{(i)}) = \frac{i}{n} \tag{17}$$

For a variable w not sampled, $w_{(i-1)} < w < w_{(i)}$, we can use a linear interpolation given by

$$F_W(w) = F_W(w_{(i-1)}) + \frac{F_W(w_{(i)}) - F_W(w_{(i-1)})}{w_{(i)} - w_{(i-1)}}(w - w_{(i-1)}) \tag{18}$$

There are three problems with the above empirical CDF when it is used in FORM. During the variable transformation, a given value of W may be beyond the range of the samples w_j $(j = 1, 2, \ldots, n)$, resulting in a breakdown in the FORM algorithm. The second problem is the nonuniformity of the samples may make the tail CDF estimation sensitive to the sample size and also cause the convergence difficulty. The last problem is that the derivative of $F_W(w)$ at some point may not exist, and this will also cause a convergence difficulty. To deal with these problems, we use the saddlepoint approximation method that follows.

3.4 Saddlepoint Approximation (SPA). The saddlepoint approximation method [25] is widely used to approximate a CDF, especially the tail of the CDF, with excellent accuracy [26,27]. The basic requirement of using SPA is to know the cumulant generating function (CGF) of the corresponding CDF. Details about SPA can be found in Ref. [28].

No closed-form CGF is available for the extreme value W, and then we generate samples for W to approximate the CGF [29]. We first estimate the moments of W, which are then converted into the corresponding cumulants. The cumulants are used to approximate the CGF. Given a set of samples w_j $(j = 1, 2, \ldots, n)$, the cumulants are computed with the following equation [30]:

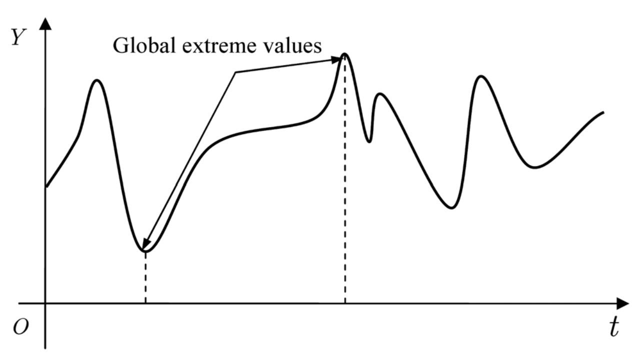

Fig. 2 A Trajectory of a stochastic process

$$\begin{cases} \kappa_1 = \dfrac{s_1}{n} \\ \kappa_2 = \dfrac{n s_2 - s_1^2}{n(n-1)} \\ \kappa_3 = \dfrac{2s_1^3 - 3n s_1 s_2 + n^2 s_3}{n(n-1)(n-2)} \\ \kappa_4 = \dfrac{-6s_1^4 + 12n s_1^2 s_2 - 3n(n-1)s_2^2}{n(n-1)(n-2)(n-3)} \\ \qquad + \dfrac{-4n(n+1)s_1 s_3 + n^2(n+1)s_4}{n(n-1)(n-2)(n-3)} \end{cases} \tag{19}$$

where s_r $(r = 1, 2, 3, 4)$ are the sums of the r-th power of the samples and are given by

$$s_r = \sum_{i=1}^{n} w_i^r \tag{20}$$

Here, we only use the first four cumulants. Based on our numerical experiments, the order is good enough. Then the CGF is approximated by [25]

$$K_W(\xi) = \sum_{i=1}^{r} \kappa_i \frac{\xi^i}{i!} \tag{21}$$

To obtain the CDF $F_W(w)$, we must solve for the saddlepoint ξ_s, which is the solution to the following equation:

$$K_W'(\xi) = w \tag{22}$$

where $K_W'(\cdot)$ is the derivative of the CGF.

The above equation can be rewritten as

$$K_W'(\xi) = \kappa_1 + \kappa_2 \frac{\xi}{1!} + \kappa_3 \frac{\xi^2}{2!} + \kappa_4 \frac{\xi^3}{3!} = w \tag{23}$$

It is a polynomial function, and its roots can be easily found.

The CDF $F_W(w)$ is then approximated with the following equations [31]:

$$F_W(w) = \Pr\{W < w\} = \Phi(z) + \phi(z)\left(\frac{1}{z} - \frac{1}{v}\right) \tag{24}$$

where

$$z = \mathrm{sign}(\xi_s)\left\{2[\xi_s w - K_W(\xi_s)]\right\}^{1/2} \tag{25}$$

$$v = \xi_s\left[K_W''(\xi_s)\right]^{1/2} \tag{26}$$

in which $\mathrm{sign}(\xi_s) = +1, -1$, or 0, depending on whether ξ_s is positive, negative, or zero. $K_W''(\cdot)$ is the second derivative of the CGF and is given by

$$K_W''(\xi_s) = \kappa_2 + \sum_{j=3}^{4} \kappa_j \frac{\xi_s^{j-2}}{(j-2)!} \tag{27}$$

SPA can give an accurate CDF estimation, especially in a tail area of the CDF. Additionally, the CDF from SPA is continuous, and it can therefore avoid the abovementioned three problems when an empirical CDF is used.

3.5 Time-Invariant Reliability Analysis. After the CDF of the extreme value of $Y(t)$ is obtained, the time-dependent limit-state function can be converted into a time-invariant function as indicated in Eq. (11). Then a time-invariant reliability method may be applied because the probability of failure is computed by

$$p_f(0, t_s) = \Pr\{g(\mathbf{X}, W) > 0\} \tag{28}$$

If FORM is used, we transform $\mathbf{X}$ and W into standard normal variables $\mathbf{U}_X$ and U_W, respectively. The transformation of W is given by

$$W = F_W^{-1}(\Phi(U_W)) \tag{29}$$

where $F_W^{-1}(\cdot)$ is the inverse function of the CDF of W. This involves the inverse saddlepoint approximation [32].

Equation (29) indicates that we need to solve for the associated saddlepoint corresponding to U_W. The saddlepoint ξ_u is obtained by solving the following equation:

$$\Phi(U_W) = \Phi(z_u) + \phi(z_u)\left(\frac{1}{z_u} - \frac{1}{v_u}\right) \tag{30}$$

where

$$z_u = \text{sign}(\xi_u)\left\{2\left[\xi_u K_W'(\xi_u) - K_W(\xi_u)\right]\right\}^{1/2} \tag{31}$$

$$v_u = \xi_u\left[K_W''(\xi_u)\right]^{1/2} \tag{32}$$

After obtaining the saddlepoint ξ_u, we have

$$W = K_W'(\xi_u) \tag{33}$$

After the transformation, the following optimization model is solved

$$\begin{cases} \min\limits_{(\mathbf{u}_W, u_W)} \|(\mathbf{u}_X, u_W)\| \\ \text{subject to } g(T(\mathbf{u}_X), T(u_W)) \geq 0 \end{cases} \tag{34}$$

The solution is the MPP and is denoted by $\mathbf{U}^*$. The reliability index β is calculated by

$$\beta = \|\mathbf{U}^*(t)\| \tag{35}$$

Then, the probability of failure is estimated by

$$p_f(0, t_s) = \Phi(-\beta) \tag{36}$$

The proposed new method has the following characteristics:

- there is no need to call the limit-state function in the simulation for $F_W(w)$
- the limit-state function $g(\cdot)$ is only called by FORM
- only one MPP search is needed

Therefore, the efficiency of the proposed new method is equal or similar to that of FORM. If we use the number of function calls to measure the efficiency, the time-dependent reliability analysis is as efficient as that of time-independent reliability analysis if FORM is employed. Previous studies have assessed the efficiency of FORM [33–35]. Their conclusions are applicable to the proposed method.

4 Examples

We now provide two examples to demonstrate the new time-dependent reliability method.

4.1 A Beam Under Time-Variant Random Loading. The first example is a beam problem adapted from Ref. [10]. The beam shown in Fig. 3 is subjected to an external force F acting at its midpoint.

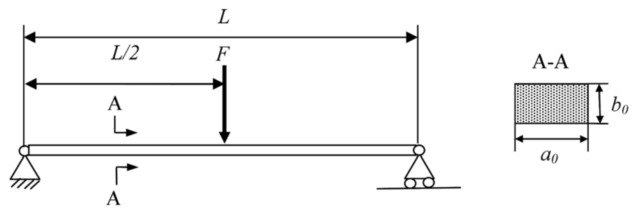

Fig. 3 A Beam under random loading

Table 1 Variables and parameters in example 1

	Mean	Std	Distribution	AC
a_0	0.2 m	0.01 m	Lognormal	N/A
B_0	0.04 m	4×10^{-3} m	Lognormal	N/A
σ_u	2.4×10^8 Pa	2.4×10^7 Pa	Lognormal	N/A
$F(t)$	4500 N	1050 N	GP	Eq. (38)
L	5 m	0	Deterministic	N/A
ρ_{st}	78.5 KN/m^3	0	Deterministic	N/A

The limit-state function is defined by

$$g(\mathbf{X}, Y(t)) = \frac{F(t)L}{4} + \frac{\rho_{st} a_0 b_0 L^2}{8} - \frac{a_0 b_0^2 \sigma_u}{4} \tag{37}$$

in which ρ_{st} is the density of steel, L is the length of the beam, a_0 and b_0 are the width and height of the cross section, respectively, and σ_u is the ultimate stress of the material. F is a Gaussian process (GP) with an autocorrelation function given by

$$\rho(t_1, t_2) = \exp\left(-\frac{(t_2 - t_1)^2}{\zeta^2}\right) \tag{38}$$

where $\zeta = 0.5$ year and is the correlation length.

Table 1 presents the variables in the limit-state function. In the table, Std and AC stand for standard deviation and autocorrelation, respectively.

In this problem, the stochastic process is the load F. It is easy to classify it as a general stress variable. Its maximum value should therefore be considered.

The time-dependent probabilities of failure over different time intervals up to [0,30] years were computed using the following three methods:

(1) The MCS: The purpose of using MCS is to assess the accuracy of the other methods. The MCS solution is regarded as the accurate solution if the number of simulations is large enough.
(2) The traditional upcrossing method (UC) with FORM [13]: It is represented by "UC" in the tables and figures of the results.
(3) The proposed method: The method is represented by "Proposed" in the tables and figures of the results.

For the proposed method, 10^4 samples were generated for estimating the CDFs of the maximum load over the time intervals. Figure 4 shows the estimated CDFs of the maximum load obtained from the proposed method. The figure indicates that the CDF curves shift to the right when t increases. This means that the right tail of the distribution becomes longer and that higher extreme load appears with a longer time interval.

The results are given in Table 2 and Fig. 5. The last two columns of Table 2 show the point solutions and their 95% confidence intervals from MCS. The solutions were resulted from 10^6 simulations. The results indicate that the traditional upcrossing

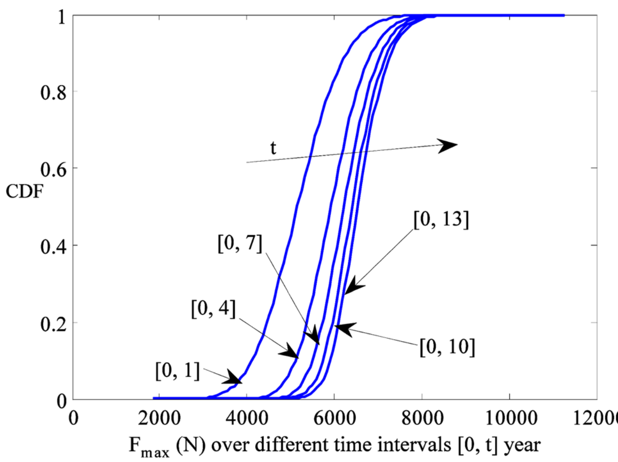

Fig. 4 CDFs of maximum load over different time intervals

Table 2 Time-dependent probabilities of failure

$[0, t_s]$ (years)	UC (10^{-3})	Proposed (10^{-3})	MCS (10^{-3})	
[0,4]	2.147	1.239	1.217	(1.215, 1.219)
[0,8]	4.079	2.004	1.989	(1.987, 1.991)
[0,12]	6.008	2.613	2.576	(2.574, 2.578)
[0,16]	7.933	3.129	3.098	(3.096, 3.100)
[0,20]	9.854	3.596	3.569	(3.567, 3.571)
[0,24]	11.771	3.980	3.956	(3.954, 3.958)
[0,28]	13.685	4.351	4.314	(4.312, 4.316)

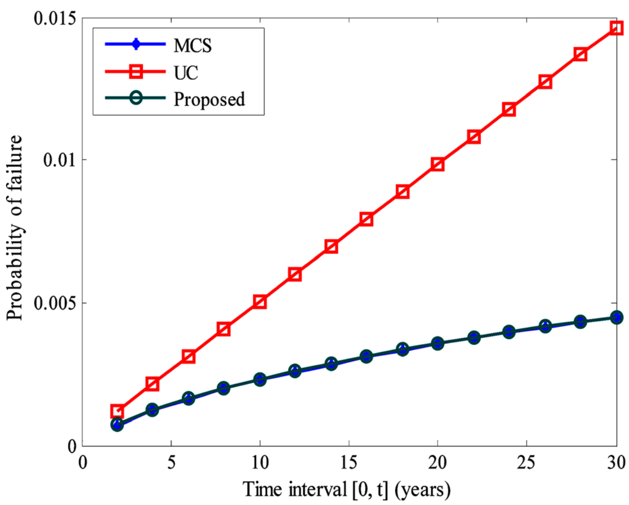

Fig. 5 Probability of failure of the beam over different time intervals

method produced large errors, especially when the probability of failure is high. This means that the upcrossing method is too conservative. The proposed method generated very accurate results.

We use the number of limit-state function calls to measure the efficiency. Table 3 gives such numbers. The table indicates that the new method is also much more efficient than the traditional upcrossing method. The reason is that the former method called the MPP search once and the latter method called the MPP search several times.

Moreover, it should be noted that even if the limit-state function of this problem in Eq. (37) is given in an explicit form, the proposed method is also applicable for more complex problems with implicit limit-state functions (black-box models).

Table 3 Number of function evaluations by the three methods

$[0, t_s]$ (years)	UC	Proposed	MCS
[0,4]	1080	50	10^6
[0,8]	1080	45	10^6
[0,12]	1080	45	10^6
[0,16]	1080	40	10^6
[0,20]	1080	40	10^6
[0,24]	1080	40	10^6
[0,28]	1080	40	10^6

4.2 Hydrokinetic Turbine Blade Under Time-Variant River Flow Loading. The stochastic process in the last example is special because its mean and standard deviation functions are constant. In this example, the stochastic process is the river velocity, which has time-variant mean, standard deviation, and autocorrelation functions. With the seasonal characteristics in the river velocity, the external load in this example is more complicated than the one in the last example. Since the river velocity governs the blade stresses, it is a general stress variable.

Figure 6 depicts the simplified cross section of the hydrokinetic turbine blade. The blade under river flow loading is shown in Fig. 7.

With a fixed tip speed ratio, the flapwise bending moment of the turbine blade is calculated as [36]

$$M_{\text{flap}} = \frac{1}{2}\rho v^2(t) C_m \tag{39}$$

in which M_{flap} stands for the flapwise bending moment, $v(t)$ is the random river flow velocity, $\rho = 1 \times 10^3 \text{kg/m}^3$ is the river flow density, and $C_m = 0.3422$ is the coefficient of moment, which is obtained from the blade element momentum theory.

Based on the historical river discharge data of the Missouri River from 1897 to 1988 at the Hermann station in Missouri [23] and the relationship between river discharge and river velocity, we fitted the mean and standard deviations of the monthly river velocity as functions of t as follows:

$$\mu_v(t) = \sum_{i=1}^{4} a_i^m \sin(b_i^m t + c_i^m) \tag{40}$$

$$\sigma_v(t) = \sum_{j=1}^{4} a_j^s \exp\left\{ -[(t - b_j^s)/c_j^s]^2 \right\} \tag{41}$$

in which a, b, and c are constants.

Similar to the wave loading [20], the monthly river velocity is assumed to be a narrowband Gaussian process (GP) with mean $\mu_v(t)$, standard deviation $\sigma_v(t)$, and autocorrelation coefficient function $\rho_v(t_1, t_2)$. The autocorrelation coefficient function $\rho_v(t_1, t_2)$ is given by

$$\rho_v(t_1, t_2) = \cos(2\pi(t_2 - t_1)) \tag{42}$$

The limit-state function is defined by

$$g = \frac{M_{\text{flap}} t_1}{EI} - \varepsilon_{\text{allow}} = \frac{\rho v(t)^2 C_m t_1}{2EI} - \varepsilon_{\text{allow}} \tag{43}$$

where $\varepsilon_{\text{allow}}$ is the allowable strain of the material, $E = 14$ GPa is the Young's modulus, and I is the moment of inertia at the root of the blade, which is computed by

$$I = \frac{2}{3} l_1 (t_1^3 - t_2^3) \tag{44}$$

in which l_1, t_1 and t_2 are the dimension variables as shown in Fig. 6. Table 4 presents the variables in this example.

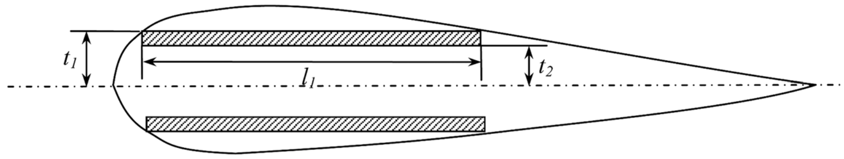

Fig. 6 Cross section at root of the turbine blade

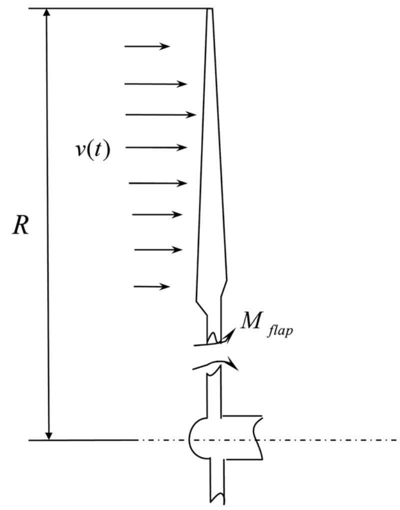

Fig. 7 River flow loading on the turbine blade

Table 4 Variables and parameters in example 2

Variable	Mean	Std	Distribution	AC
v (m/s)	$\mu_v\ (t)$	$\sigma_v\ (t)$	GP	In Eq. (42)
l_1 (m)	0.22	2.2×10^{-3}	Gaussian	N/A
t_1(m)	0.025	2.5×10^{-4}	Gaussian	N/A
t_2 (m)	0.019	1.9×10^{-4}	Gaussian	N/A
$\varepsilon_{\text{allow}}$	0.025	2.5×10^{-4}	Gaussian	N/A

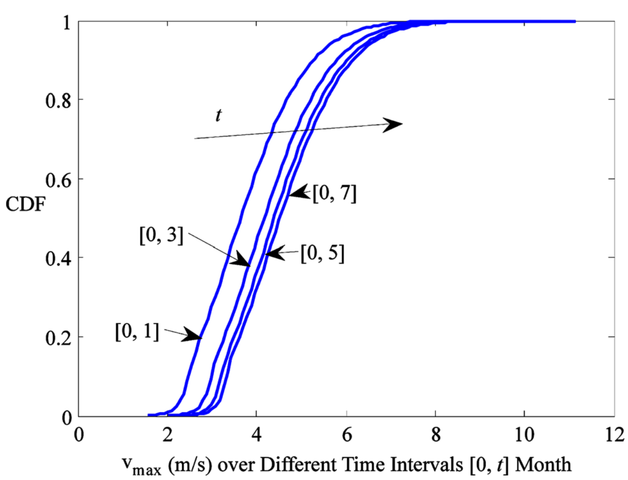

Fig. 8 CDFs of maximum river velocity over different time intervals

We computed the probabilities of failure of the hydrokinetic turbine blade over different time intervals up to [0,12] months using MCS, UC, and the proposed method. The numbers of performed simulations and generated samples are also the same as those in example 1.

Table 5 Time-dependent probabilities of failure

$[0, t_s]$ (months)	UC (10^{-3})	Proposed (10^{-3})	MCS (10^{-3})	
[0,4]	0.088	0.085	0.080	(0.079, 0.080)
[0,5]	1.664	1.128	1.115	(1.113, 1.117)
[0,6]	2.366	1.185	1.157	(1.155, 1.159)
[0,7]	4.886	2.667	2.817	(2.815, 2.819)
[0,8]	5.072	2.667	2.817	(2.815, 2.819)
[0,9]	5.118	2.667	2.817	(2.815, 2.819)

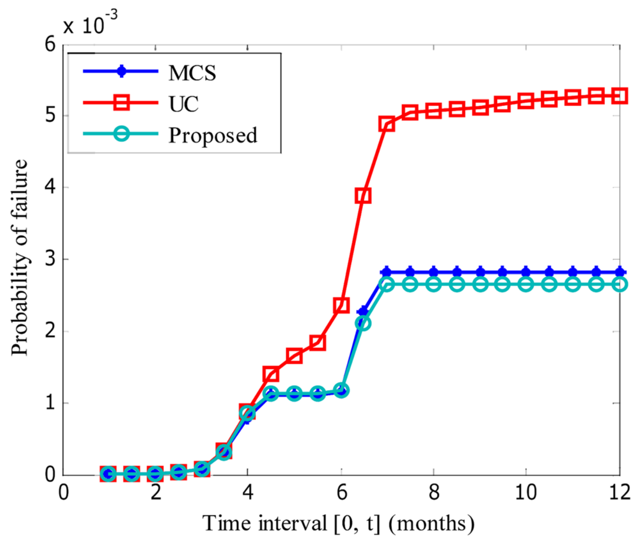

Fig. 9 Probability of failure of the hydrokinetic turbine blade

Table 6 Number of function evaluations

$[0, t_s]$ (months)	UC	Proposed	MCS
[0,4]	2522	102	10^6
[0,5]	6539	85	10^6
[0,6]	6761	121	10^6
[0,7]	11500	98	10^6
[0,8]	6678	98	10^6
[0,9]	19399	98	10^6

Figure 8 presents the estimated CDFs of the maximal river velocity over different time intervals obtained from the proposed method.

Table 5 and Fig. 9 show the probabilities of failure of the turbine blade over different time intervals obtained from the three methods. The 95% confidence intervals of the MCS solutions are also given in the last column of Table 5.

The results show that the proposed method is much more accurate than the traditional upcrossing rate method. The results also demonstrate that the proposed method is applicable for time-dependent reliability analysis with a nonstationary stochastic process. The new method is also more efficient than the traditional method as indicated by the number of function calls in Table 6.

5 Conclusions

A new sampling method to time-dependent reliability method has been proposed for limit-state functions that are implicit with respect to time and are functions of a general strength or stress stochastic process. The method employs a sampling approach to estimating distributions of the extreme value of the stochastic process over the time interval under consideration. The distribution is estimated by saddlepoint approximations. The extreme value is then used to replace its corresponding stochastic process. Then the time-dependent problem becomes a time-invariant problem, and a time-invariant reliability analysis method can be used. In this work, we used FORM.

The new method has advantages over the traditional upcrossing method in the following two aspects:

- The number of evaluations of the limit-state function is significantly reduced. The method is therefore much more efficient.
- The accuracy of reliability analysis has also been improved significantly.

In addition to FORM, other reliability methods can also be used after the time-dependent problem has been transformed into a time-invariant one.

The proposed method is limited to the limit-state functions that are implicit with respect to time. The input stochastic process should be either a general strength variable or a general stress variable.

Our future work includes incorporating the proposed method into reliability-based design optimization, using the idea to more general limit-state functions, and investigating the possible extension of the method to problems with multiple stochastic processes.

Acknowledgment

This material is based upon work supported in part by the Office of Naval Research through contract ONR N000141010923 (Program Manager – Dr. Michele Anderson), the National Science Foundation through Grant No. CMMI 1234855, and the Intelligent Systems Center at the Missouri University of Science and Technology.

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