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Accelerated Life Testing (ALT) Design Based on Computational Reliability Analysis

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Accelerated life testing (ALT) design is usually performed based on assumptions of life distributions, stress–life relationship, and empirical reliability models. Time-dependent reliability analysis on the other hand seeks to predict product and system life distribution based on physics-informed simulation models. This paper proposes an ALT design framework that takes advantages of both types of analyses. For a given testing plan, the corresponding life distributions under different stress levels are estimated based on time-dependent reliability analysis. Because both aleatory and epistemic uncertainty sources are involved in the reliability analysis, ALT data is used in this paper to update the epistemic uncertainty using Bayesian statistics. The variance of reliability estimation at the nominal stress level is then estimated based on the updated time-dependent reliability analysis model. A design optimization model is formulated to minimize the overall expected testing cost with constraint on confidence of variance of the reliability estimate. Computational effort for solving the optimization model is minimized in three directions: (i) efficient time-dependent reliability analysis method; (ii) a surrogate model is constructed for time-dependent reliability under different stress levels; and (iii) the ALT design optimization model is decoupled into a deterministic design optimization model and a probabilistic analysis model. A cantilever beam and a helicopter rotor hub are used to demonstrate the proposed method. The results show the effectiveness of the proposed ALT design optimization model. Copyright © 2015 John Wiley & Sons, Ltd.

Keywords: accelerated life testing (ALT); reliability analysis; Bayesian theory; uncertainty; optimization

1. Introduction

Engineering products are commonly tested at higher than nominal stress conditions to yield failures in a relatively short time. This method is referred to as accelerated life testing (ALT),[1] which is an essential part of the reliability assurance process. A reasonably designed ALT plan can not only reduce the cost of reliability testing, but also significantly accelerate the product development cycle. ALT design [2] usually refers to designing the optimal stress testing levels and numbers of tests allocated to each stress level; this is commonly based on assumptions of life distributions at different stress levels and relationship between stress and life.

Several approaches have been developed during the past decades to design optimal ALT plans. For instance, Dorp and Mazzuchi developed a general Bayesian inference model for ALT design by assuming that the failure times at each stress level are exponentially distributed.[3] They also developed a general Bayes–Weibull inference model for ALT by assuming the failure times follow Weibull distribution.[4] Elsayed and Zhang[5] developed a multiple-stress ALT model to overcome the limitation of traditional ALT models that only focus on a single stress. Zhang and Meeker[6] presented Bayesian methods for ALT planning with one accelerating variable and discussed how to obtain the optimal testing plan. Lee and Pan studied the parameter estimation method of step-stress ALT (SSALT) model.[7] Voiculescu et al.[8] studied the Arrhenius–Exponential model of ALT techniques using the maximum likelihood (ML) and Bayesian methods. Even though many methods have been developed and studied, most of these methods purely depend on testing data and make assumptions about the life distribution and stress–life relationship. Physics-informed computational models, however, are seldom considered during ALT design.

With the same ultimate purpose of predicting product reliability, model-based reliability analysis methods based on physics-informed models [9] have been intensively studied by another group of researchers in the past decades. Especially in recent years, reliability analysis based on computer simulations is becoming increasingly popular for systems where testing is unaffordable. Such model-based reliability analysis has been successfully applied to the prediction of reliability of automobiles,[10] aircrafts,[11] offshore structures,[12] civil structures,[13] and many other problems[14] by propagating uncertainty through computer simulation models. In order to investigate the degradation of product reliability over time, a group of time-dependent reliability analysis methods have been developed recent years. Examples of time-dependent reliability analysis methods include the PHI2 method developed by Andrieu-Renaud et al.[15], the probability density evolution method,[16] the upcrossing rate-based method,[17] the composite limit-state

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function method,[18] the first-order sampling approach,[19] and importance sampling approaches proposed by Mori and Ellingwood,[20] Dey and Mahadevan,[21] and Singh and Mourelatos.[22] Because time-dependent reliability is connected to ALT through the relationship between life and reliability, integration of physics-informed time-dependent reliability analysis models and ALT methods is a promising way of designing optimal testing plans to reduce product development cost and time.

Efforts have been reported earlier to integrate ALT design with reliability analysis using computational models. For example, Zhang and Mahadevan developed an integration method by combing the prior computational prediction and test data.[23] Zhang and Mahadevan also discussed how to update the life-distribution using testing data [24] in fatigue testing. These methods mainly focus on the updating of life distribution based on ALT. In this paper, we develop a new ALT design optimization model based on time-dependent reliability analysis. Instead of making assumptions about the stress–life relationship, physics-informed modeling and simulation are used to represent the underlying relationship between stress and life.

The contributions of this paper are summarized as: (i) Formulation of a new ALT design optimization model, which enables the decision maker to use the physics-informed computational model to guide the ALT design optimization. (ii) Relaxation of assumptions on life distribution and stress–life relationship in ALT design. (iii) Introduction of methods to efficiently solve the new ALT design optimization model.

In Section 2, the concept of ALT design is reviewed. Following that, in Section 3, time-dependent reliability analysis based on physics-informed simulations is discussed. Section 4 proposes the new ALT design method based on time-dependent reliability analysis and discusses how to solve the optimization model. Two examples of reliability testing design are given in Section 5. Conclusions are drawn in Section 6.

2. ALT design

ALT is used to predict the reliability of a product under nominal stress level in a reasonable timeframe. The prediction is performed based on the stress–life relationship obtained from reliability testing at stress levels that are higher than the nominal stress level. The 'stress' here refers to any factor that can be used in ALT to reduce the time to failure. It can be temperature, force, voltage, humidity, etc. As shown in Figure 1, reliability tests are performed at higher stress levels, from which experimental data of life are collected. Reliability at the nominal stress level is then estimated based on the experimental data at higher stress levels.

In order to predict the reliability of a product at the nominal stress level based on experimental data at higher stress levels, it is commonly assumed in the literature that the life distribution at any stress level follows either a Lognormal or Weibull distribution. Based on this distribution assumption, distribution parameters at tested stress levels are estimated. The distribution parameters are then fitted as a function of stress level in the logarithmic coordinates (as indicated in Figure 1). The commonly used fitting functions include linear, quadratic, and exponential functions. For instance, when a linear function is used to fit the mean of life, the mean function is given by

$$\mu_T(S) = \alpha_1 + \alpha_2 \log(S) \tag{1}$$

in which S is the stress level, $\mu_T(S)$ is the mean of life distribution at stress level S, and α_1 and α_2 are two parameters estimated based on the experimental data.

Similarly, the other parameters of the life distribution are fitted using functions or are assumed to be constant over different stress levels. Based on the functions of parameters with respect to stress level, distribution parameters at the nominal stress level S_0 are obtained. The product reliability R_{S_0} at the nominal stress level is then estimated based on the assumed life distribution type and the estimated distribution parameters.

Because the experimental data may not be enough to exactly estimate the parameters of the function given in Eq. (1), the function parameters are uncertain, which results in uncertainty in the estimation of R_{S_0}. In addition, the assumption of Lognormal distribution or Weibull distribution may not hold. This also contributes the uncertainty of R_{S_0}. Motivated by reducing the variation in the estimation

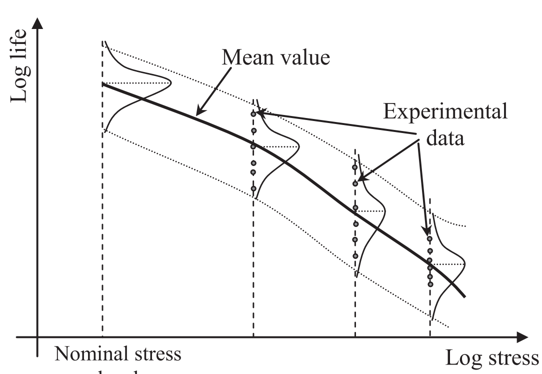

Figure 1. Stress–life relationship in ALT

of R_{S_0} subjected to constraints on testing resources, ALT design methods have been proposed during the past decades. For instance, Pascual and Montepiedra proposed a model-robust ALT planning method based on a weighted asymptotic sample ratio (ASR).[25] Pascual also discussed a methodology for the designing of ALT plan robust to misspecification of the stress–life relationship.[26] Fard and Li developed a step stress ALT model to optimize the hold time of step stress tests.[27] Liu and Tang proposed a Bayesian design optimization model to minimize the expected pre-posterior variance of reliability prediction.[28] They also developed a design optimization approach for ALT with an auxiliary acceleration factor (AAF).[29]

All the above reviewed ALT methods have the same objective, which is to minimize the expected variance of reliability prediction and subject to constraint on testing budget. Most of these empirical methods, however, rely on assumptions and do not utilize the information of physics models, which are often available for computer simulations. In the subsequent sections, we will first introduce the method of life prediction using time-dependent reliability analysis, which is based on physics computational model of the system. Based on that, we present a new ALT design framework.

3. Life distribution based on time-dependent reliability analysis

3.1. Physics-based reliability analysis

The fundamental principle of physics-based reliability analysis is to predict the probability of failure based on the joint probability density function (PDF) of input variables and the relationship between input variables and response obtained from physics models. For a given realization of random variables, the response is predicted using numerical computational models such as finite element analysis (FEA), computational fluid dynamics (CFD), or other physics simulation models. Based on that, the probability of failure is computed as

$$p_f = \int_{g(\mathbf{X}) \leq 0} f_X(\mathbf{x}) d\mathbf{x} \tag{2}$$

where $\mathbf{X}$ *is a vector of random variables,* $g(\mathbf{X})$ is the response function, $f_X(\mathbf{x})$ is the joint PDF, and $g(\mathbf{X}) \leq 0$ is the failure event.

Many reliability analysis methods have been proposed during the past decades to efficiently and accurately solve the above integration. The most widely used methods include the first-order reliability method (FORM), second-order reliability method (SORM), and different methods of Monte Carlo sampling.

3.2. Time-dependent reliability analysis with only aleatory uncertainty

After considering time-dependent factors, the response function becomes $G(t) = g(\mathbf{X}, \mathbf{Y}(t), t)$, where $\mathbf{Y}(t)$ is a vector of stochastic processes and t is time. The time-dependent probability of failure or the first-passage probability of failure over a time interval of interest $[0, t]$ is given by

$$p_f(0, t) = \Pr\{G(\tau) = g(\mathbf{X}, \mathbf{Y}(\tau), \tau) \leq 0, \exists \tau \in [0, t]\} \tag{3}$$

where '∃' means 'there exists'.

Eq. (3) can also be written as

$$F_T(t) = \Pr\{G(\tau) = g(\mathbf{X}, \mathbf{Y}(\tau), \tau) \leq 0, \exists \tau \in [0, t]\} \tag{4}$$

in which $F_T(t)$ is the cumulative density function (CDF) of life distribution.

Assuming that the statistics of the random variables $\mathbf{X}$ and stochastic processes $\mathbf{Y}(t)$ are precisely known and there is no error in the response function $g(\mathbf{X}, \mathbf{Y}(t), t)$, the time-dependent probability of failure (i.e. CDF of a given life t) can be estimated using the time-dependent reliability analysis methods. When the system response is a monotonic function with respect to time, time-dependent reliability analysis is equivalent to time-instantaneous reliability analysis at the initial or last time instant. When the response function is not a monotonic function, a group of time-dependent reliability analysis methods have been proposed in the past decades.[30] Among these methods, the outcrossing rate method based on the Rice's formula is the most widely used.[30] In the outcrossing rate method, the outcrossing events are assumed to be independent and following the Poisson distribution. Based on this assumption, $p_f(0, t)$ or $F_T(t)$ is estimated by

$$p_f(0, t) = 1 - R(0) \exp\left\{-\int_0^t v^+(\tau) d\tau\right\} \tag{5}$$

where $R(0)$ stands for the reliability at the initial time instant and $v^+(\tau)$ is the outcrossing rate at time instant τ. $v^+(\tau)$ is estimated by

$$v^+(\tau) \approx \lim_{\Delta t \to 0} \frac{\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau), \tau) > 0 \cap g(\mathbf{X}, \mathbf{Y}(\tau + \Delta t), \tau + \Delta t) \leq 0\}}{\Delta t} \tag{6}$$

where '∩' stands for 'intersection'.

The outcrossing rate method is accurate for most problems when the failure threshold is high. When the failure threshold is low, the Poisson assumption regarding the outcrossing rate may result in significant error in the reliability estimate. In order to improve the accuracy of reliability analysis, other methods have been developed by releasing the Poisson assumption, such as the composite limit-state function method,[18] the stochastic discretization method,[31] and first-order sampling approach (FOSA).[19] In FOSA, $F_T(t)$ up to the time interval of interest can be obtained from one analysis. In this paper, FOSA is therefore employed for problems with non-monotonic limit state functions.

The basic principle of FOSA is to model the system response as a random process based on the following probability equivalency.

$$F_T(t) = \Pr\{G(\tau) = g(\mathbf{X},\ \mathbf{Y}(\tau),\ \tau) \le 0,\ \exists\ \tau \in [0,\ t]\} \approx \Pr\{L_G(\tau) \le 0,\ \exists\ \tau \in [0,\ t]\} \tag{7}$$

in the above equation, $L_G(\tau)$ is a Gaussian random process. Based on the modeling of $L_G(\tau)$, $F_T(t)$ is estimated through simulation. The three main steps of FOSA are briefly summarized as below.

(*a*). *Surrogate model construction*: Training points $\boldsymbol{\tau} = [\tau_1,\ \tau_2,\ \cdots,\ \tau_{train}]$ of time are generated over [0, t] using design of experiments (DOE) methods. For each training point $\tau \in \boldsymbol{\tau}$, the associated most probable point (MPP), $\mathbf{u}^*(\tau)$, is identified by solving the following optimization model

$$\begin{cases} \min\ \beta(\tau) = \|\mathbf{u}(\tau)\| \\ \mathbf{u}(\tau) = [\mathbf{u_X},\ \mathbf{u_Y}(\tau)] \\ G(\tau) = g(T(\mathbf{u_X}),\ T(\mathbf{u_Y}(\tau)),\ \tau) \le 0 \end{cases} \tag{8}$$

in which $\beta(\tau)$ is the reliability index at τ, $\|\cdot\|$ is the determinant of a vector, $\mathbf{u_X}$ and $\mathbf{u_Y}(\tau)$ are vectors of standard normal variables, $T(\cdot)$ is an operator, which transforms $\mathbf{u_X}$ and $\mathbf{u_Y}(\tau)$ into original random variables $\mathbf{X}$ and $\mathbf{Y}(\tau)$.

After solving Eq. (8) for each chosen time value τ, we have $\boldsymbol{\tau} = [\tau_1,\ \tau_2,\ \cdots,\ \tau_{train}]$ and $\boldsymbol{\beta} = [\beta(\tau_1),\ \beta(\tau_2), \cdots,\ \beta(\tau_{train})]$. We also obtain training points for the correlation function of $L_G(\tau)$ as follows

$$\rho_L(\tau_i,\ \tau_j) = \boldsymbol{\alpha}_\mathbf{X}(\tau_i)\boldsymbol{\alpha}_\mathbf{X}^T(\tau_j) + \boldsymbol{\alpha}_\mathbf{Y}(\tau_i)\boldsymbol{\rho}(\tau_i,\ \tau_j)\boldsymbol{\alpha}_\mathbf{Y}^T(\tau_j),\quad \forall \tau_i,\ \tau_j \in [\tau_1,\ \tau_2,\ \cdots,\ \tau_{train}] \tag{9}$$

where $\boldsymbol{\alpha}_\mathbf{X}(\tau_k) = \mathbf{u}_\mathbf{X}^*(\tau_k)/\|\mathbf{u}^*(\tau_k)\|$, $\boldsymbol{\alpha}_\mathbf{Y}(\tau_k) = \mathbf{u}_\mathbf{Y}^*(\tau_k)/\|\mathbf{u}^*(\tau_k)\|$, $k = i,\ j$, and $\boldsymbol{\rho}(\tau_i,\ \tau_j)$ is given by

$$\boldsymbol{\rho}(\tau_i,\ \tau_j) = \begin{bmatrix} \rho_{Y_1}(\tau_i,\ \tau_j) & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_{Y_s}(\tau_i,\ \tau_j) \end{bmatrix}_{s \times s} \tag{10}$$

in which $\rho_{Y_k}(\tau_i,\ \tau_j)$, $k = 1,\ 2, \cdots,\ s$, are the autocorrelation coefficients of $\mathbf{U}_{Y_k}(t)$ between time instants τ_i and τ_j, and s is the number of stochastic processes.

With the training points, $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, and $\rho_L(\tau_i,\ \tau_j)$, $\forall\,\tau_i,\ \tau_j \in [\tau_1,\ \tau_2,\ \cdots,\ \tau_{train}]$, surrogate models $\hat{\beta}(t)$ and $\hat{\rho}_L(t_i,\ t_j)$ are then constructed using the Kriging method reviewed in Appendix A.

(*b*). *Random process modeling*: As discussed in Eq. (7), the original response can be approximated using an equivalent random process $L_G(\tau)$. After discretizing the time interval of interest into h time instants, t_i, $i = 1,\ 2,\ \cdots,\ h$, $L_G(\tau)$ is simulated using the expansion optimal linear estimation method (EOLE) method [32] as below:

$$L_G \approx -\beta(\tau) + \sum_{i=1}^{r} \frac{\xi_i}{\sqrt{\eta_i}} \boldsymbol{\varphi}_i^T \boldsymbol{\rho}_{L\tau}(\tau),\qquad \forall \tau \in [0,\ t] \tag{11}$$

where ξ_i, $i = 1,\ 2,\ \cdots,\ r$, is a vector of independent standard normal variables, $\boldsymbol{\rho}_{L\tau}(\tau) = [\rho_L(\tau,\ t_1),\ \rho_L(\tau,\ t_2),\ \cdots, \rho_L(\tau,\ t_h)]^T$, η_i and $\boldsymbol{\phi}_i^T$ are the eigenvalues and eigenvectors of correlation matrix $\boldsymbol{\rho}_L$, $r \le h$ is the number of terms of expansion, and $\boldsymbol{\rho}_L$ is obtained by inputting time instants, t_i, $i = 1,\ 2,\ \cdots,\ h$, into $\widehat{\rho}_L(t_i,\ t_j)$. Note that the eigenvalues η_i are sorted in decreasing order.

(*c*). *Reliability analysis*: With the expression given in Eq. (11), samples of $L_G(\tau)$ are generated by discretizing [0, t] into W time instants and generating N samples for each random variable of ξ_i. After that, a sampling matrix is obtained as follows

$$\widetilde{L}_{N \times W} = \begin{pmatrix} l(t_1,\ 1) & l(t_2,\ 1) & \cdots & l(t_W,\ 1) \\ l(t_1,\ 2) & l(t_2,\ 2) & \cdots & l(t_W,\ 2) \\ \vdots & \vdots & \ddots & \vdots \\ l(t_1,\ N) & l(t_2,\ N) & \cdots & l(t_W,\ N) \end{pmatrix}_{N \times W}. \tag{12}$$

The global minimum values of $\widetilde{L}_{N \times W}$ are then identified by

$$l_{\min,\ i} = \min\{l(t_1,\ i),\ l(t_2,\ i),\ \cdots,\ l(t_W,\ i)\}, i = 1,\ 2,\ \cdots,\ N \tag{13}$$

$F_T(t)$ or $p_f(0, t)$ is then approximated as

$$p_f(0, t) = \frac{\sum_{i=1}^{N} I^{-}(l_{\min, i})}{N} \tag{14}$$

where $I^{-}(l_{\min, i}) = 1$ if $l_{\min, i} \leq 0$, otherwise, $I^{-}(l_{\min, i}) = 0$.

More details about time-dependent reliability analysis can be found in.[19] After performing time-dependent reliability analysis at different life values using Eq. (4), CDF and PDF of life distribution (as shown in Figure 2) are obtained.

3.3. *Time-dependent reliability analysis with both aleatory and epistemic uncertainty*

3.3.1. Epistemic uncertainty. It is often difficult to obtain enough data to fully characterize the statistics of the input random variables and stochastic processes. In addition, there are multiple approximation errors in computer simulation models. Because of both reasons, epistemic uncertainty, which refers to uncertainty because of lack of knowledge, affects the time-dependent reliability estimate. The epistemic uncertainty can be roughly classified into two groups.

- *Data Uncertainty*: Parameters of random variables and stochastic processes are usually modeled based on the collected data. Limited data, noises in measurement, and sensor degradation result in uncertainty in the parameters.
- *Model Uncertainty*: Computer simulation models are widely used in reliability analysis. These simulation models will inevitably have some errors because of model form assumptions and numerical approximations.

Modeling of epistemic uncertainty sources has been extensively studied during the past decades.[33] In this paper, we mainly focus on their effects on time-dependent reliability analysis.

3.3.2. Effects of epistemic uncertainty on time-dependent reliability. Let θ be a vector of variables representing various sources of epistemic uncertainty, which include not only the epistemic parameters of random variables and stochastic processes, but also epistemic uncertainty of model form error and numerical errors. Estimated distributions of θ can be obtained using interval analysis, fuzzy set, and Bayesian theory based on limited data or expert opinions.[33] Denote the PDF of θ as $f(\theta)$. After introducing θ into the limit state function of time-dependent reliability analysis, the response function given in Section 3.2 becomes $G(t) = g(\mathbf{X}, \mathbf{Y}(t), \theta, t)$, where $\mathbf{X}$ and $\mathbf{Y}(t)$ include random variables and stochastic processes those with known parameters and those with epistemic parameters, θ. For any given realization of θ, the reliability analysis problem reduces to a time-dependent reliability analysis problem with only aleatory uncertainty. As shown in Figure 3, different life distributions will be obtained from time-dependent reliability analysis under different realizations of θ. From different life distributions, different reliability estimates are obtained with respect to the life of interest. The reliability estimate under epistemic uncertainty θ is therefore uncertain (as indicated in Figure 3). Time-dependent reliability analysis with aleatory and epistemic uncertainty is computationally intensive. An efficient reliability analysis method has been developed in Ref. [34] to reduce the computational cost. Because the focus of this paper is ALT design, details of time-dependent reliability analysis with aleatory and epistemic uncertainty are not given here; see Ref. [34] for details.

We already introduced background concepts on ALT design and life distribution analysis using time-dependent reliability analysis. In the following section, we will discuss how to perform ALT design based on time-dependent reliability analysis.

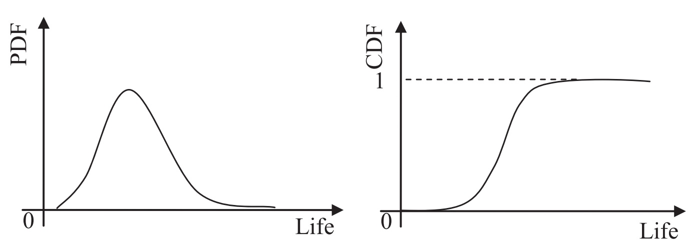

Figure 2. Illustration of life distribution from reliability analysis with only aleatory uncertainty

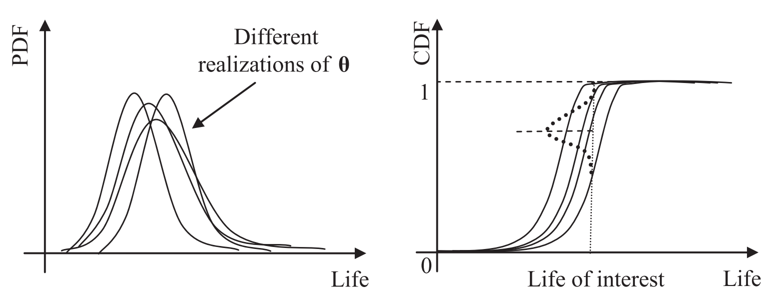

Figure 3. Life distribution from reliability analysis with both aleatory and epistemic uncertainty

4. ALT design based on time-dependent reliability analysis

In this section, we first analyze the connection between ALT and time-dependent reliability analysis. Based on that, the ALT design optimization problem is formulated next. We then discuss how to solve the ALT design optimization problem.

4.1. *Connection between ALT and time-dependent reliability analysis*

As presented in Sections 2 and 3, ALT predicts the product reliability based on experimental data, whereas time-dependent reliability analysis estimates reliability based on physics-based computational simulation models. These two methods are connected by their ultimate goal: reliability prediction at the nominal stress level. After introducing stress level S into the limit-state function of time-dependent reliability analysis, the response function given in Section 3.3 is rewritten as

$$G(S,\ t) = g(\mathbf{X},\ \mathbf{Y}(t),\ \theta,\ t,\ S). \tag{15}$$

For a given realization of θ, the time-dependent probability of failure given in Eq. (4) becomes

$$F_T(S,\ t)|\theta = \Pr\{G(S,\ \tau) = g(\mathbf{X},\ \mathbf{Y}(\tau),\ \theta,\ \tau,\ S) \le 0,\ \exists\, \tau \in [0,\ t]\}. \tag{16}$$

The above equation predicts the probability that the product life is less than t under stress level S for a given realization of epistemic uncertainty, θ. If time-dependent reliability analysis is performed at multiple stress levels, as indicated in Figure 4, similar to ALT, a stress–life relationship will be obtained.

The difference between the stress–life relationships obtained from ALT and time-dependent reliability analysis is that the stress–life relationship is modeled using some assumed functions (i.e. linear, quadratic functions) based on experimental data in ALT whereas the life distributions at different stress levels are connected through the underlying physics model in time-dependent reliability analysis. Because ALT and time-dependent reliability analysis aim to model the same stress–life relationship, physics-informed time-dependent reliability analysis model can be used to guide the data-based ALT design by fusing the information from these two sources.

4.2. *ALT design optimization model*

4.2.1. Formulation of ALT design optimization model. In ALT design, decision makers hope to minimize the reliability testing cost while maintaining the confidence of product reliability prediction. Based on this motivation, we defined the following objective and constraints for ALT design optimization.

(a) Optimization objective

The reliability testing costs consist of two parts: (i) manufacturing cost of testing units, and (ii) inspection, labor, and other costs related to testing time. The first cost is only related the number of testing units. The second cost is related to not only the number of testing units but also the expected testing time. The optimization objective is therefore defined as follows

$$C_{total} = \sum_{i=1}^{m} n_i (C_1 + C_2 t_{test}(S_i)) \tag{17}$$

where C_{total} is the total expected testing cost, m is the number of stress levels in ALT (for example, $m = 2$ for two-level stress testing design and $m = 3$ for three-level stress testing design), $\mathbf{n} = [n_1,\ n_2,\ \cdots,\ n_m]$ and $\mathbf{S} = [S_1,\ S_2,\ \cdots,\ S_m]$ are design variables, n_i is the number of tests allocated to stress level S_i, $t_{test}(S_i)$ is the expected testing time per testing unit under stress level S_i, C_1 is the manufacturing cost per testing unit, and C_2 is the testing cost per unit testing time (i.e. labor cost and inspection cost).

(b) Optimization constraints

The optimization constraints relate to the number of tests allocated to each stress level, the testing stress levels, and confidence of reliability estimation. For the constraint of reliability estimate confidence, the most common method is to use the expected pre-posterior variance given as [6]

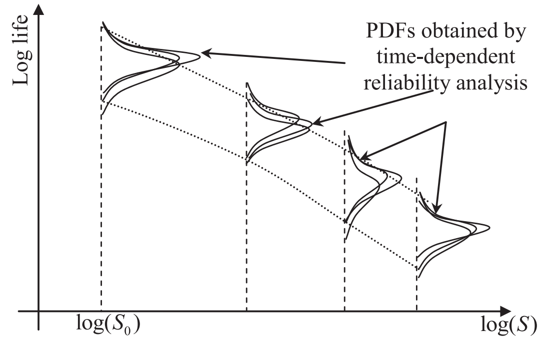

Figure 4. Illustration of stress–life relationship obtained from time-dependent reliability analysis

$$V(R_{S_0}) = E_{\mathbf{t}|\ \mathbf{n},\ \mathbf{S}}\left(Var_{\hat{\boldsymbol{\theta}}|\mathbf{t},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})\right) \tag{18}$$

where S_0 is the nominal stress level, R_{S_0} is the prediction of product reliability at the nominal stress level, $\hat{\boldsymbol{\theta}}$ are the posterior samples of θ after Bayesian updating using data $\mathbf{t}$, $Var_{\hat{\boldsymbol{\theta}}|\mathbf{t},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})$ is the pre-posterior variance of R_{S_0} under given values of $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{S}$, $E_{\mathbf{t}|\mathbf{n},\ \mathbf{S}}(\cdot)$ is the expectation of pre-posterior variance by considering variation in $\mathbf{t}$, and variation in $\mathbf{t}$ is obtained from the marginal distribution of $\mathbf{t}$.

The pre-posterior variance $Var_{\hat{\boldsymbol{\theta}}|\mathbf{t},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})$ has been widely used for ALT design in previous studies.[6,28] By assuming that the posterior distribution follows a multivariate normal distribution, the pre-posterior variance can be estimated using the expected Fisher information matrix [35], whose elements are second derivatives of the logarithmic likelihood. Because the expected Fisher information matrix is only related to the ML estimate (MLE) $\overline{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ for given life testing data $\mathbf{t}$, the variation of $Var_{\hat{\boldsymbol{\theta}}|\mathbf{t},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})$ comes from the variation in $\overline{\boldsymbol{\theta}}$. Zhang and Meeker [6] showed that the prior probability distribution of θ provides an excellent approximation to the distribution of $\overline{\boldsymbol{\theta}}$ because the distribution of $\overline{\boldsymbol{\theta}}$ will converges to $\boldsymbol{\theta}$ when sample size increases. This implies that the variation of $Var_{\hat{\boldsymbol{\theta}}|\mathbf{t},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})$ mainly comes from the variation in $\boldsymbol{\theta}$. Eq. (18) is therefore rewritten as

$$V(R_{S_0}) = E_{\boldsymbol{\theta}}\left(Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})\right). \tag{19}$$

The above equation requires multi-dimensional integration over the probability domain of $\boldsymbol{\theta}$. For some problems, it might be computationally expensive because the estimation of $Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})$ based on time-dependent reliability analysis is computationally intensive. In this paper, we use the probabilistic constraint given below to substitute for the expectation constraint:

$$\Pr_{\theta}\left\{g(\mathbf{n},\ \mathbf{S},\ \boldsymbol{\theta}) = Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{S_0}) > e\right\} \leq [\Pr] \tag{20}$$

where [Pr] is a probability specified by decision maker to quantify the desired confidence of reliability prediction and e is the threshold for pre-posterior variance of R_{S_0}.

There are two main advantages for the use of a probabilistic constraint instead of expectation. First, as shown in Figure 5, $Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{S_0})$ is always larger than zero and follows a distribution with a long right tail. By controlling the percentile value at the right tail using the probabilistic constraint, the confidence of reliability prediction can be easily guaranteed. Using an expectation constraint, however, may still have a high probability that the variance of reliability prediction is large. Second, the computational effort for evaluation of the expectation of $E_{\theta}\left(Var_{\hat{\theta}|\theta,\ \mathbf{n},\ \mathbf{S}}(R_{S_0})\right)$ is typically more intensive than that for the approximation of a single percentile value.

(c) Optimization model

Based on the objective and constraints, the ALT design optimization model is formulated as

$$\begin{aligned}
&\min_{\mathbf{n},\mathbf{S}}\quad C_{total} = \sum_{i=1}^{m} n_i(C_1 + C_2 t_{test}(S_i)) \\
&s.t.\quad \Pr_{\boldsymbol{\theta}}\left\{g(\mathbf{n},\ \mathbf{S},\ \boldsymbol{\theta}) = Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{S_0}) > e\right\} \leq [\Pr] \\
&\qquad 1 \leq \mathbf{n} \leq n_{\max} \\
&\qquad S_0 \leq \mathbf{S} \leq S_{\max} \\
&\qquad \sum_{i=1}^{m} n_i \leq n_{\max} \\
&\qquad n_{i-1} < n_i,\ \text{ for } i = 1,\ 2,\ \cdots,\ m
\end{aligned} \tag{21}$$

in which $n_{\max}$ is the maximum number of tests that can be allocated to each stress level and $S_{\max}$ is the maximum allowable testing stress.

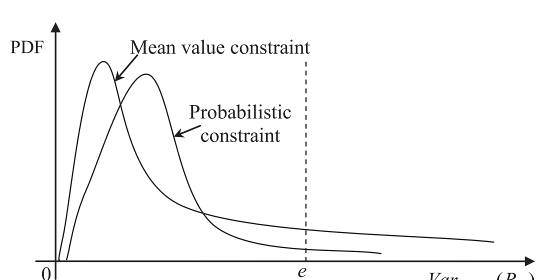

Figure 5. Illustration of distribution of $Var_{\hat{\theta}|\theta,\ \mathbf{n},\ \mathbf{S}}(R_{S_0})$

In the above optimization model, C_1 and C_2 are obtained from field data, [Pr] and e are specified by the decision maker according to product reliability requirement, and $t_{test}(S_i)$ and $Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{s_0})$ are two unknown terms. We will discuss how to model these two unknown terms in the subsequent sections.

4.2.2. Modeling of $t_{test}(S_i)$

Before performing ALT, we do not have information about the expected testing time at different stress levels. In this paper, we use time-dependent reliability analysis to estimate the expected testing time. At a stress level S_i, the expected testing time, $t_{test}(S_i)$, is computed by

$$t_{test}(S_i) = \iint t f_t(t|\boldsymbol{\theta},\ S_i) f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) dt d\boldsymbol{\theta} \tag{22}$$

where $f_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ is the joint PDF of $\boldsymbol{\theta}$ and $f_t(t|\boldsymbol{\theta},\ S_i)$ is the PDF of t given $\boldsymbol{\theta}$ and S_i, which is given by

$$f_t(t|\boldsymbol{\theta},\ S_i) = \frac{F_T(t+\Delta t)|\boldsymbol{\theta},\ S_i - F_T(t)|\boldsymbol{\theta},\ S_i}{\Delta t} = \frac{\Delta \Pr}{\Delta t} \tag{23}$$

and

$$\begin{aligned}\Delta \Pr = {} & \Pr\{G(S_i,\ \tau) = g(\mathbf{X},\ \mathbf{Y}(\tau),\ \theta,\ \tau,\ S_i) \le 0,\ \exists \tau \in [0,\ t+\Delta t]\} \\ & - \Pr\{G(S_i,\ \tau) = g(\mathbf{X},\ \mathbf{Y}(\tau),\ \theta,\ \tau,\ S_i) \le 0,\ \exists \tau \in [0,\ t]\}.\end{aligned} \tag{24}$$

The above equation is solved based on time-dependent reliability analysis at stress level S_i. Substituting Eq. (22) into the optimization model given in Eq. (21), the optimization model is rewritten as

$$\begin{aligned}\min_{\mathbf{n},\mathbf{S}}\ \ & C_{total} = \sum_{i=1}^{m} n_i \left[C_1 + C_2 \iint t f_t(t|\boldsymbol{\theta},\ S_i) f_\theta(\boldsymbol{\theta}) dt d\boldsymbol{\theta}\right] \\ s.t.\ \ & \Pr_{\boldsymbol{\theta}}\left\{g(\mathbf{n},\ \mathbf{S},\ \boldsymbol{\theta}) = Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{s_0}) > e\right\} \le [\Pr] \\ & 1 \le \mathbf{n} \le n_{\max} \\ & S_0 \le \mathbf{S} \le S_{\max} \\ & \sum_{i=1}^{m} n_i \le n_{\max} \\ & n_{i-1} < n_i,\ \text{ for } i = 1,\ 2,\ \cdots,\ m.\end{aligned} \tag{25}$$

Considering that the optimizer will evaluate the objective function numerous times when solving the above optimization model and the integration operator in the objective will call $f_t(t|S_i,\ \boldsymbol{\theta})$ many times, we construct a surrogate model for $f_t(t|S,\ \boldsymbol{\theta})$ to save the computational effort. As indicated in Eq. (23), $f_t(t|S_i,\ \boldsymbol{\theta})$ is computed from $F_T(t)|\boldsymbol{\theta},\ S_i$ which is obtained from time-dependent reliability analysis; we therefore construct a surrogate model for $F_T(t)|\boldsymbol{\theta},\ S_i$ and compute $f_t(t|S_i,\ \boldsymbol{\theta})$ using Eq. (23) based on the surrogate model. The Kriging approach briefly summarized in Appendix A is used in this paper to construct the surrogate model.

4.2.3. Evaluation of pre-posterior variance of reliability, $Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{s_0})$

Because $E_{\mathbf{t}|\ \mathbf{n},\ \mathbf{S}}\left(Var_{\hat{\boldsymbol{\theta}}|\mathbf{t},\ \mathbf{n},\ \mathbf{S}}(R_{s_0})\right)$ is transformed into $E_{\boldsymbol{\theta}}\left(Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{s_0})\right)$ and $\boldsymbol{\theta}$ provides an approximation to the MLE $\bar{\boldsymbol{\theta}}$ (as discussed in Section 4.2.1), $\boldsymbol{\theta}$ mentioned in this section is also $\bar{\boldsymbol{\theta}}$. For a given value of $\boldsymbol{\theta}$ (i.e. $\bar{\boldsymbol{\theta}}$), the reliability under nominal stress level R_{s_0} is predicted by inputting $\boldsymbol{\theta}$, S_0, the life of interest t_e into the surrogate model $\hat{F}_T(t,\ \boldsymbol{\theta},\ S)$. The asymptotic variance of R_{s_0} is computed based on the first-order Taylor expansion as

$$Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{s_0}) = \left[\frac{\mathbf{d}R_{s_0}}{\mathbf{d}\boldsymbol{\theta}}\right] \mathbf{C} \left[\frac{\mathbf{d}R_{s_0}}{\mathbf{d}\boldsymbol{\theta}}\right]' \tag{26}$$

where $\left[\frac{\mathbf{d}R_{s_0}}{\mathbf{d}\boldsymbol{\theta}}\right] = \left[\frac{dR_{s_0}}{d\theta_1},\ \frac{dR_{s_0}}{d\theta_2},\ \cdots,\ \frac{dR_{s_0}}{d\theta_q}\right]$, q is the number of epistemic uncertain variables, and $\mathbf{C}$ is the covariance matrix of $\boldsymbol{\theta}$ in the posterior distribution.

In this paper, we use the expected Fisher information matrix to estimate $\mathbf{C}$. $\mathbf{C}$ is given by

$$\mathbf{C} = [\mathbf{I}(\boldsymbol{\theta})]^{-1} \tag{27}$$

where $\mathbf{I}(\theta)$ *is the expected Fisher information matrix with* (*i*, *j*)-th element as

$$I_{ij}(\boldsymbol{\theta}) = -\sum_{k=1}^{m} n_k E_{\boldsymbol{\theta}}^{k}\left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log(L(t_k|\boldsymbol{\theta}))\right],\quad i,\ j = 1,\ 2,\ \cdots,\ q. \tag{28}$$

$E_{\boldsymbol{\theta}}^{k}\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log(L(t_k|\boldsymbol{\theta}))\right]$ is the expectation of $\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log(L(t_k|\boldsymbol{\theta}))$ by considering the variation of t_k at the k-th testing stress level. It is computed as

$$E_{\boldsymbol{\theta}}^{k}\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log(L(t_k|\boldsymbol{\theta}))\right]=\int\frac{\partial^2\log(L(t_k|\boldsymbol{\theta}))}{\partial\theta_i\partial\theta_j}f_t(t_k|\boldsymbol{\theta},\ S_k)dt_k. \tag{29}$$

The term $\frac{\partial^2\log(L(t_k|\boldsymbol{\theta}))}{\partial\theta_i\partial\theta_j}$ is computed using a second-order numerical partial differentiation method. As mentioned in Section 4.2.2, $f_t(t_k|\boldsymbol{\theta},\ S_k)$ is obtained based on the surrogate model $\hat{F}_T(t,\ \boldsymbol{\theta},\ S)$ (constructed in Section 4.2.2) by fixing values of θ and S_k. For a given value t_k, $L(t_k|\theta)$ is computed as

$$L(t_k|\boldsymbol{\theta})=\frac{\hat{F}_T(t_k+\Delta t,\ \boldsymbol{\theta},\ S_k)-\hat{F}_T(t_k,\ \boldsymbol{\theta},\ S_k)}{\Delta t}. \tag{30}$$

4.2.4. Solution of the design optimization model. The ALT formulation in Eq. (21) is a nonlinear optimization problem with mixed integer (**n**) and continuous (**S**) design variables and probabilistic constraint. Directly solving Eq. (21) results in a double-loop procedure. The inner loop is probabilistic analysis and outer loop is design optimization. Because the probabilistic analysis needs to solve $g(\mathbf{n},\ \mathbf{S},\ \boldsymbol{\theta})$ repeatedly, the double-loop procedure is computationally very expensive. In order to improve the efficiency, we first transform the uncertain variables, $\boldsymbol{\theta}$, from the original space to an equivalent standard normal space ($\mathbf{u}_{\boldsymbol{\theta}}$) and rewrite Eq. (21) as

$$\begin{aligned}
&\min_{\mathbf{n},\mathbf{S}}\ \ C_{total}=\sum_{i=1}^{m}n_i[C_1+C_2t_{test}(S_i)]\\
&s.t.\ \ \Pr_{\mathbf{u}_\theta}\left\{g(\mathbf{n},\ \mathbf{S},\ T(\mathbf{u}_{\boldsymbol{\theta}}))=Var_{\hat{\boldsymbol{\theta}}|T(\mathbf{u}_{\boldsymbol{\theta}}),\ \mathbf{n},\ \mathbf{S}}(R_{S_0})>e\right\}\le[\Pr]\\
&\qquad 1\le\mathbf{n}\le n_{\max}\\
&\qquad S_0\le\mathbf{S}\le S_{\max}\\
&\qquad \sum_{i=1}^{m}n_i\le n_{\max}\\
&\qquad n_{i-1}<n_i,\ \text{for}\ i=1,\ 2,\ \cdots,\ m
\end{aligned} \tag{31}$$

where $T(\mathbf{u}_{\boldsymbol{\theta}})$ is a transformation operator which transforms $\mathbf{u}_{\boldsymbol{\theta}}$ into $\boldsymbol{\theta}$.

The formulation in Eq. (31) helps to decouple the two loops. The sequential optimization and reliability assessment (SORA) method [36] is employed to decouple the optimization model (Eq. (31)) into a deterministic design optimization model and a probabilistic analysis model. The deterministic design optimization model is given by

$$\begin{aligned}
&\min_{\mathbf{n},\mathbf{S}}\ \ C_{total}=\sum_{i=1}^{m}n_i[C_1+C_2t_{test}(S_i)]\\
&s.t.\ \ g\left(\mathbf{n},\ \mathbf{S},\ T\left(\mathbf{u}_\theta^*\right)\right)=Var_{\hat{\boldsymbol{\theta}}|T(\mathbf{u}_{\boldsymbol{\theta}}^*),\ \mathbf{n},\ \mathbf{S}}(R_{S_0})\le e\\
&\qquad 1\le\mathbf{n}\le n_{\max}\\
&\qquad S_0\le\mathbf{S}\le S_{\max}\\
&\qquad \sum_{i=1}^{m}n_i\le n_{\max}\\
&\qquad n_{i-1}<n_i,\ \text{for}\ i=1,\ 2,\ \cdots,\ m
\end{aligned} \tag{32}$$

where $\mathbf{u}_\theta^*$ *is obtained from the following probabilistic analysis model*

$$\begin{aligned}
&\max_{\mathbf{u}_{\boldsymbol{\theta}}^*}\ \ g\left(\mathbf{n},\ \mathbf{S},\ T\left(\mathbf{u}_\theta^*\right)\right)\\
&s.t.\ \ \left\|\mathbf{u}_{\boldsymbol{\theta}}^*\right\|=\beta_0
\end{aligned} \tag{33}$$

in which $\beta_0=\boldsymbol{\Phi}(1-[\Pr])$.

The deterministic design optimization model given in Eq. (32) is a mixed integer and continuous optimization problem, and requires a mixed integer programming algorithm. In this work, the genetic algorithm [37] is used to solve Eq. (32). Algorithms for solving above probabilistic analysis model in (Eq. (33)) are well established and available in.[38] Based on Eqs. (32) and (33), Eq. (31) can be solved in a decoupled procedure as given in Figure 6.

In the above figure, ε is a convergence criterion for the optimization. In the next sub-section, we will summarize the overall procedure of ALT design based on time-dependent reliability analysis.

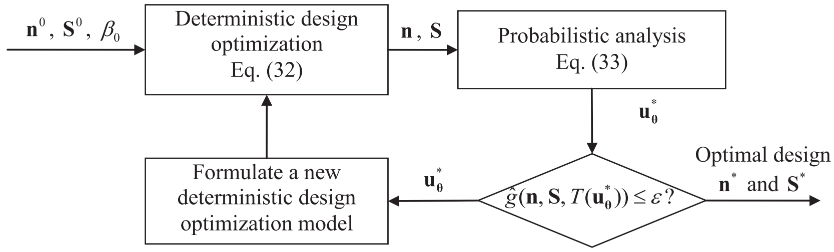

Figure 6. Flowchart for solving the ALT design optimization model

4.3. *Implementation procedure*

The overall numerical procedure of the proposed ALT design optimization is summarized in Figure 7. There are mainly five steps:

- Step 1: Generate training points for t, $\boldsymbol{\theta}$, and S using DOE and according to the maximum allowable testing stress.
- Step 2: Perform time-dependent reliability analysis at the training points. If the system response is a monotonic function with respect to time, only time-instantaneous reliability analysis is needed. Otherwise, time-dependent reliability analysis needs to be employed.
- Step 3: Construct a surrogate model for $\hat{F}_T(t,\ \boldsymbol{\theta},\ S)$ using the training points. If the accuracy (i.e. bias and variance) of the surrogate model output is not satisfactory, add more training points until the desired accuracy is achieved.
- Step 4: Model the expected testing time using the method presented in Section 4.2.2 and the surrogate model obtained from Step 3.
- Step 5: Solve the ALT design optimization model using the method given in Section 4.2.4.

5. Numerical example

5.1. *A cantilever beam with an initial random fatigue crack*

5.1.1. Problem statement A cantilever beam (shown in Figure 8) with an initial random planar fatigue crack (through crack) near the fixed support is employed as the first example. The beam is subjected to a cyclic loading at the end of the beam. The fatigue crack length is assumed to grow vertically downward because of the cyclic loading. Failure is defined as when the fatigue crack length is larger than 0.15 inch. An ALT plan needs to be designed to estimate the reliability of the beam.

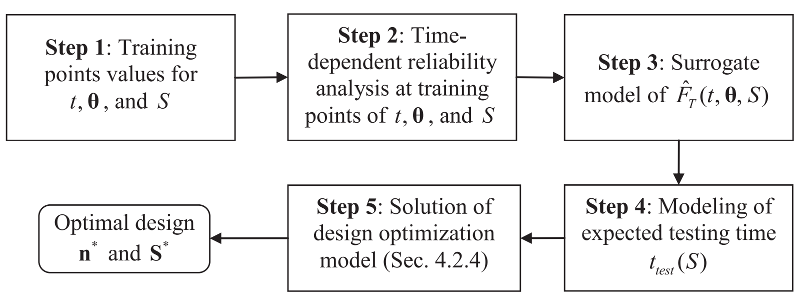

Figure 7. Overall numerical procedure of ALT design optimization

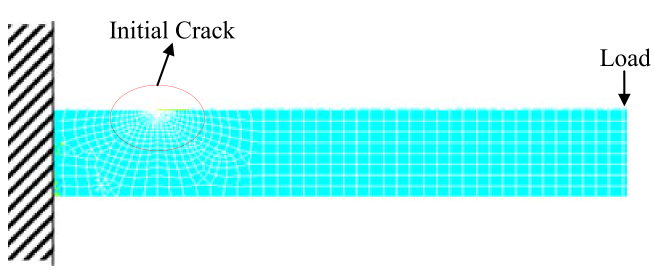

Figure 8. Finite element analysis model of the beam

In order to numerically predict the fatigue crack growth, the Paris law [39] is used to estimate the rate of crack growth during each cycle as follows

$$\frac{da}{dN} = C(\Delta K)^m \tag{34}$$

where a is the crack length, *N* is the number of cycles, *C* and *m* are model parameters of Paris law, and $\Delta K = K(a, S)$ is the stress intensity factor, which is a function of current crack geometry and applied load.

The uncertain variables and parameters of the crack growth model are summarized in Table I. Based on available information, the distribution of initial crack length a_0 is assumed to be precisely known, while the Paris law parameters (*C* and *m*) are assumed to have Gaussian distribution but with unknown means and standard deviations.

The lack of knowledge (epistemic uncertainty) about the mean and standard deviation of *C* and *m* is expressed through prior distributions given in Table II. Let the manufacturing cost per testing unit be $C_1 = \$500$, and the testing cost per cycle be $C_1 = \$0.1$. Assume that the maximum number of tests that can be allocated to a stress level is 80, the nominal stress level is 2.5 N, and the allowable testing stress is 15 N. The ALT design optimization model is thus formulated as

$$\begin{aligned} &\min_{\mathbf{n},\mathbf{S}} \quad C_{total} = \sum_{i=1}^{m} n_i\left(C_1 + C_2 t_{test}(S_i)\right) \\ &s.t. \quad \Pr_{\theta}\left\{g(\mathbf{n},\ \mathbf{S},\ \boldsymbol{\theta}) = Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{s_0}) > e\right\} \leq [\Pr] \\ &\qquad 1 \leq \mathbf{n} \leq 80 \\ &\qquad 2.5 \leq \mathbf{S} \leq 15 \\ &\qquad \sum_{i=1}^{m} n_i \leq 80 \\ &\qquad n_{i-1} < n_i, \quad \text{for } i = 1,\ 2,\ \cdots,\ m \end{aligned} \tag{35}$$

where $\boldsymbol{\theta} = [\mu_C,\ \sigma_C,\ \mu_m,\ \mu_m]$, $e = 0.001$, and $[\Pr] = 0.05$.

For given realizations of epistemic parameters $\boldsymbol{\theta} = [\mu_C,\ \sigma_C,\ \mu_m,\ \mu_m]$, the fatigue reliability of the beam is computed as

$$R_{s_0} = \Pr\{a_{N_e}(C,\ m,\ a_0) < 0.15\} \tag{36}$$

in which a_{N_e} *is the crack length after* N_e cycles of loading and N_e is the life of interest ($N_e = 1.5 \times 10^4$ cycles). a_{N_e} is solved using Eq. (34).

Because the crack length grows with time, the response function is a monotonic function of time. The time-dependent reliability analysis therefore reduces to time-instantaneous reliability analysis at the last time instant (N_e).

In the following sections, we will solve the ALT design optimization model using the proposed method and perform ALT design for two cases, namely two-stress level ALT design and three-stress level ALT design, respectively.

5.1.2. Expected testing time under different stress levels Based on the surrogate model, $\hat{F}_T(t,\ \boldsymbol{\theta},\ S)$, we analyzed the expected testing time under different stress levels. Figure 9 gives the relationship between the expected testing time and stress level. It shows that the expected testing time decreases significantly with the stress level. However, after the stress level increases to a particular value (larger than 10 N), the reduction in the expected testing time is not significant anymore.

Table I. Uncertainty variables in the fatigue reliability analysis of the beam

Variable	Mean	Standard deviation	Distribution
a_0	0.03	0.005	Gaussian
C	Unknown	Unknown	Gaussian
m	Unknown	Unknown	Gaussian

Table II. Information of uncertain distribution parameters

Variable	Mean	Standard deviation	Distribution
μ_C	3.2115×10^{-8}	2×10^{-9}	Gaussian
σ_C	2.5×10^{-9}	2×10^{-11}	Gaussian
μ_m	2.1277	0.1	Gaussian
σ_m	0.15	0.005	Gaussian

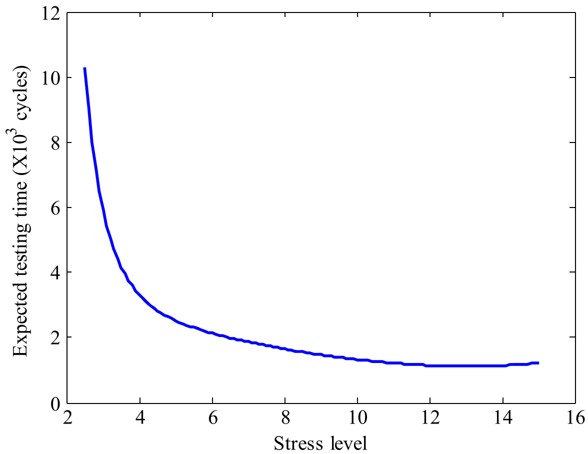

Figure 9. Expected testing cycles under different stress levels

5.1.3. Effect of number of tests on the variance of reliability prediction Based on the model given in Section 4.2.3, we used the two-stress level testing design case to investigate the effect of the number of tests on the variance of reliability prediction. We first fix θ at the mean values of their prior distributions and stress levels at 7 N and 10 N. We then compute the posterior variances of reliability prediction at different combinations of numbers of tests. As indicated in Figure 10, the variance in reliability prediction can be reduced by increasing the number of tests. This demonstrates the effectiveness of the proposed posterior variance prediction model.

5.1.4. ALT design optimization We solved the ALT design optimization model for the two-stress level and three-stress level testing designs, respectively. The design variables are the number of tests at each stress level, and the magnitude of each stress level. Thus, there are four design variables ($\mathbf{n} = [n_1,\ n_2]$, $\mathbf{S} = [S_1,\ S_2]$) for the two-stress level testing design and six design variables ($\mathbf{n} = [n_1,\ n_2,\ n_3]$, $\mathbf{S} = [S_1,\ S_2,\ S_3]$) for the three-stress level testing design. Table III gives the optimized design variables and corresponding total expected testing costs for the two-stress level case. Following that, Table IV presents the iteration history of the design optimization. It shows that the decoupled optimization approach in Eqs. (32) and (33) is very efficient.

Similar to the two-stress level case, Tables V and VI present the optimal solution and iteration history for the three-stress level ALT design case.

The results show that the required numbers of tests in the two-stress level testing are higher than those of three-stress level testing. For this example, the expected testing cost of two-stress level testing is also higher than that of three-stress level testing. This

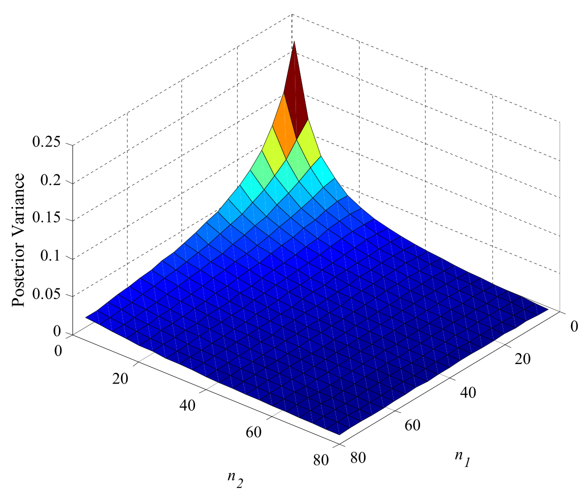

Figure 10. Variance of reliability prediction under different numbers of tests (two-stress level)

Table III. Optimization results of two-stress level ALT design

Variable	n_1	n_2	S_1 (N)	S_2 (N)	C_{total}
Optimal value	24	20	9.98	8.64	\$ 2.83×10^4

Table IV. Iteration history of the two-stress level ALT design

Iteration	n_1	n_2	S_1 (N)	S_2 (N)	$\hat{g}(\mathbf{n}, \mathbf{S}, T(\mathbf{u}_\theta^*))$
1	20	16	9.60	8.50	0.0120
2	24	20	9.98	8.64	1.57×10^{-4}

Table V. Optimization results of three-stress level ALT design

Variable	n_1	n_2	n_3	S_1 (N)	S_2 (N)	S_3 (N)	C_{total}
Optimal value	14	8	14	9.45	8.18	10.61	\$ 2.30×10^4

Table VI. Iteration history of the three-stress level ALT design

Iteration	n_1	n_2	n_3	S_1 (N)	S_2 (N)	S_3 (N)	$\hat{g}(\mathbf{n}, \mathbf{S}, T(\mathbf{u}_\theta^*))$
1	5	8	1	9.21	6.79	9.10	0.0437
2	15	7	14	10.09	8.66	11.12	0.0042
3	14	8	14	9.45	8.18	10.61	7.18×10^{-4}

is an interesting result, showing that with fewer stress levels, more tests at each level are required to achieve the same confidence in the reliability prediction when compared to the case with more stress levels, while such a result is qualitatively expected, the proposed approach combines physics-based modeling and time-dependent reliability analysis to quantify the number of tests required at each level. This is an important benefit of the proposed methodology.

5.2. *Helicopter rotor hub*

5.2.1. Problem statement A helicopter rotor hub yoke made of laminated composite material is adopted from Ref. [24] as our second example. The laminate is tapered as shown in Figure 11 and the resin pockets are possible sites for initiating fatigue delamination failure.

The reliability of the composite rotor hub yoke with respect to the delamination failure is required to be evaluated. The delamination life is desired to be at least $N_e = 3\times10^3$ cycles. A limit state function has been established based on non-linear FEA, virtual crack-closure technique (VCCT), and available experimental data.[24] The limit state function is modeled using response surface and is given by

$$g = G_{crit} - 0.175344\times\gamma\times\left(0.569 - 0.0861\ E_{11} + 0.0231P - 0.117\theta - 0.000546\ P^2 + 0.00376\ \theta^2 + 0.0046\ P\ \theta\right) \tag{37}$$

where θ is the force angle, P is the load with standard deviation of 23.35 kN, E_{11} is the modulus of composite material which follows normal distribution with unknown mean $\mu_{E_{11}}$ and standard deviation $\sigma_{E_{11}}$, and G_{crit} is the critical value of the strain energy release rate, which is fitted as a function of load cycles based on experimental data as follows [24]

$$G_{crit} = \beta_0 + \beta_1 \log(N) + \varepsilon \tag{38}$$

in which β_0 and β_0 are two parameters obtained from experiment, N is the number of load cycles, ε is regression residual used to account for model uncertainty in G_{crit}, and γ is an epistemic uncertain parameter to represent model uncertainty in computational simulation model. Same as Ref. [24], γ is assumed to be follow a uniform distribution between [0.9, 1.1]. The time-dependent reliability of the rotor hub is computed by

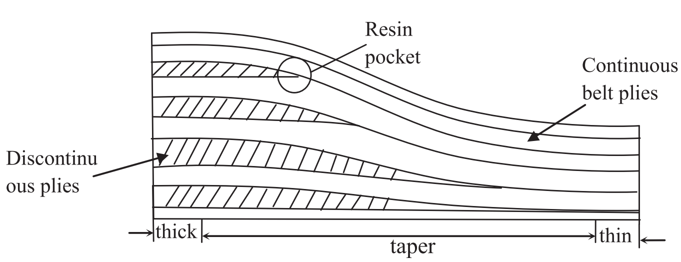

Figure 11. Half of the symmetric section of tapered composite in a helicopter rotor hub

Table VII. Uncertainty variables in the helicopter rotor hub example

Variable	Mean value	Standard deviation	Distribution	Uncertainty type
θ (degree)	10	2	Gaussian	Aleatory
P (kN)	Testing stress	23.35	Gaussian	Aleatory
β_0	448.56	20	Gaussian	Aleatory
β_1	-58.57	4	Gaussian	Aleatory
$\mu_{E_{11}}$ (10^9 N/m^2)	6.9	0.2	Gaussian	Epistemic
$\sigma_{E_{11}}$ (10^9 N/m^2)	0.09	0.003	Gaussian	Epistemic
ε	0	36.6	Gaussian	Epistemic
γ	1	$0.2/2\sqrt{3}$	Uniform	Epistemic

Table VIII. Optimization results of two-stress level ALT design

Variable	n_1	n_2	S_1	S_2	C_{total}
Optimal value	100	9	298.16	320.02	\$ 1.16×0^5

Table IX. Optimization results of three-stress level ALT design

Variable	n_1	n_2	n_3	S_1	S_2	S_3	C_{total}
Optimal value	17	14	47	143.87	179.18	247.78	\$ 9.57×10^4

$$R(0,\ N) = \Pr\{g(\tau) > 0,\ \forall\tau\in[0,\ N]\}. \tag{39}$$

Table VII summarizes the aleatory and epistemic uncertain parameters and variables of this example.

Based on the above information, the ALT design optimization model for the rotor hub is formulated as

$$\begin{aligned}
&\min_{\mathbf{n},\mathbf{S}}\ C_{total} = \sum_{i=1}^{m} n_i(C_1 + C_2 t_{test}(S_i)) \\
&s.t.\ \Pr_{\theta}\left\{g(\mathbf{n},\ \mathbf{S},\ \boldsymbol{\theta}) = Var_{\hat{\boldsymbol{\theta}}|\boldsymbol{\theta},\ \mathbf{n},\ \mathbf{S}}(R_{s_0}) > e\right\} \le [\Pr] \\
&\qquad 1 \le \mathbf{n} \le 100 \\
&\qquad 125 \le\ \mathbf{S} \le 320 \\
&\qquad n_{i-1} < n_i,\ \text{ for } i = 1,\ 2,\ \cdots,\ m
\end{aligned} \tag{40}$$

where $C_1 = \$1000$, $C_2 = \$1.5\times 10^{-4}$, $\theta = [\mu_E,\ \sigma_E,\ \varepsilon,\ \gamma]$, $e = 2\times 10^{-4}$, and $[\Pr] = 0.02$.

5.2.2. ALT design optimization Tables VIII and IX show the ALT design optimization results obtained using the proposed method for the two-stress and three-stress level cases. Similar conclusions are obtained as Example 1. Note that the overall cost of two-level testing is higher than that of three-level testing, similar to Example 1.

6. Conclusion

ALT is widely used in manufacturing to accelerate the product development process and guarantee the reliability of the product. Optimization of the ALT plan is vital to reduce the product testing time and cost. Conventional ALT design methods only depend on testing data and are based on assumptions of life distributions at different stress levels and relationship between stress and life. For some problems, it is possible for us predict the reliability based on physics-informed computational simulation models. Integration of ALT with the physics-informed reliability analysis methods will bridge the gap between ALT design and probabilistic engineering design (reliability analysis) for such problems.

A novel ALT design optimization model is developed in this paper based on time-dependent reliability analysis. The expected testing cost is minimized with constraints on the variance of reliability prediction and testing parameters. For a given testing plan, the reliability of the product at different stress levels is analyzed using physics-based time-dependent reliability analysis. Because there are epistemic uncertainty sources involved in the reliability analysis based on computational models, ALT data are used to reduce the epistemic uncertainty in the reliability estimate using Bayesian statistics. The ALT plan is then optimized such that the variance of reliability prediction from the computational model can satisfy the requirement specified by the decision maker. Directly solving the proposed ALT design model is very computationally expensive. Therefore a surrogate modeling method and a decoupled optimization approach are employed to reduce the computational cost. A cantilever beam with fatigue crack growth and a composite laminate helicopter rotor hub with fatigue delamination are used to demonstrate the effectiveness of the proposed method.

Future research may investigate the application of the proposed methodology to realistic, complicated problems. The numerical examples mainly focus on time-dependent fatigue reliability problem. Other time-dependent reliability cases will be also considered in our future work. This study only considers reliability with respect to a single criterion. Future work needs to extend this approach to include multiple failure modes and system with multiple components. In such realistic cases, the different failure modes may grow at different rates, and different tests even at the same test level might cause the components to fail in different sequences. Extension of the proposed ALT design approach to such situations will make the methodology powerful for realistic applications.

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Appendix A Kriging model

In a Kriging model, the output is assumed to be a Gaussian stochastic process.[40,41] The Kriging model of an unknown function $g(\mathbf{x})$ with inputs of $\mathbf{x}$ is given by [40]

$$\hat{g}(\mathbf{x}) = \mathbf{h}(\mathbf{x})^T \upsilon + \varepsilon(\mathbf{x}) \tag{41}$$

where $\upsilon = [\upsilon_1,\ \upsilon_2,\ \cdots,\ \upsilon_p]^T$ is a vector of unknown coefficients, $\mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x}),\ h_2(\mathbf{x}),\ \cdots,\ h_p(\mathbf{x})]^T$ is a vector of regression functions, $\mathbf{h}(\mathbf{x})^T\upsilon$ is the trend of prediction, and $\varepsilon(\mathbf{x})$ is usually assumed to be a Gaussian process with zero mean and covariance $Cov[\varepsilon(\mathbf{x}_i),\ \varepsilon(\mathbf{x}_j)]$.

The covariance between two points $\mathbf{x}_i$ and $\mathbf{x}_j$ is given by

$$Cov\left[\varepsilon(\mathbf{x}_i),\ \varepsilon\left(\mathbf{x}_j\right)\right] = \sigma_\varepsilon^2 R\left(\mathbf{x}_i,\ \mathbf{x}_j\right) \tag{42}$$

in which σ_ε^2 *is the process variance and* $R(\cdot,\ \cdot)$ is the correlation function. A variety of correlation functions have been studied in the literature, such as the exponential function, Gaussian function, cubic function, and spline function.

With n_s training points, $[\mathbf{x}_i,\ g(\mathbf{x}_i)]_{i=1,\ 2,\ \cdots,\ n_s}$, the coefficients υ are obtained by

$$\upsilon = \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{g} \tag{43}$$

where $\boldsymbol{R}$ *is the correlation matrix with element,* $R(\mathbf{x}_i,\ \mathbf{x}_j)$, $i,\ j = 1,\ 2,\ \cdots,\ n_s$, $\mathbf{H} = \left[\mathbf{h}(\mathbf{x}_1)^T,\ \mathbf{h}(\mathbf{x}_2)^T,\ \cdots,\ \mathbf{h}(\mathbf{x}_{n_s})^T\right]^T$, and $\mathbf{g} = [g(\mathbf{x}_1),\ g(\mathbf{x}_2),\ \cdots,\ g(\mathbf{x}_{n_s})]^T$.

For a new point $\mathbf{x}$, the expected value of the prediction is given by

$$\hat{g}(\mathbf{x}) = \mathbf{h}(\mathbf{x})^T\upsilon + \mathbf{r}(\mathbf{x})^T\mathbf{R}^{-1}(\mathbf{g} - \mathbf{H}\upsilon) \tag{44}$$

where

$$\mathbf{r}(\mathbf{x}) = [R(\mathbf{x},\ \mathbf{x}_1),\ R(\mathbf{x},\ \mathbf{x}_2),\ \cdots,\ R(\mathbf{x},\ \mathbf{x}_{n_s})] \tag{45}$$

The mean square error (MSE) of the prediction is given by [42]

$$\begin{aligned} \mathrm{MSE}(\mathbf{x}) = \sigma_\varepsilon^2\{&1 - \mathbf{r}(\mathbf{x})^T\mathbf{R}^{-1}\mathbf{r}(\mathbf{x}) \\ &+ \left[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{r}(\mathbf{x}) - \mathbf{h}(\mathbf{x})\right]^T\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\left[\mathbf{H}^T\mathbf{R}^{-1}\mathbf{r}(\mathbf{x}) - \mathbf{h}(\mathbf{x})\right] \end{aligned} \tag{46}$$

in which

$$\sigma_\varepsilon^2 = \frac{(\mathbf{g} - \mathbf{H}\upsilon)^T\mathbf{R}^{-1}(\mathbf{g} - \mathbf{H}\upsilon)}{n_s}. \tag{47}$$

The unknown parameters a_k, $k = 1,\ 2,\ \cdots,\ n$ can be estimated by maximizing the log likelihood given as

$$\ln[(\mathbf{g}|\mathbf{R})] = \frac{n_s \ln \sigma_\varepsilon^2}{2} - \frac{\ln|\mathbf{R}|}{2} \tag{48}$$

where $|\boldsymbol{R}|$ *is the determinant of* $\boldsymbol{R}$. *More details about Kriging model method can be found in,*[40,41] *and a MATLAB Kriging toolbox is available.*[42]

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