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Product lifetime cost is largely determined by product lifetime reliability. In product design, the former is minimized while the latter is treated as a constraint and is usually estimated by statistical means. In this work, a new lifetime cost optimization model is developed where the product lifetime reliability is predicted with computational models derived from physical principles. With the physics-based reliability method, the state of a system is indicated by computational models, and the time-dependent system reliability is then predicted for a given set of distributions and stochastic processes in the model input. A sampling approach to extreme value distributions of input stochastic processes is employed to make the system reliability analysis efficient and accurate. The physics-based reliability analysis is integrated with the lifetime cost model. The integration enables the minimal lifetime costs including those of maintenance and warranty. Two design examples are used to demonstrate the proposed model.

Keywords: product design; lifetime cost; reliability

1. Introduction

High reliability is expected for almost all products. Reliability is the probability that a product performs its intended function under specified conditions for a specified period. It is important to address reliability issues in the early design stages where many design concepts are generated and then selected. During the design conceptualization, many tools can be used to identify potential failure modes, their causes, their consequences, and the possible ways to eliminate these failures or reduce their likelihood and consequences. These tools include failure modes and effects analysis (FMEA), fault tree analysis (FTA) and quality function deployment (QFD).

Later in the parameter design stage, there are two major tasks regarding reliability. One is the reliability analysis (Rackwitz 2001; Du and Sudjianto 2004; Mahadevan and Smith 2006; Guo and Du 2010) where the reliability is predicted, and the other is reliability-based design optimization (RBDO) (Du, Sudjianto, and Huang 2005; Du and Huang 2007; Du 2008; Du, Guo, and Beeram 2008) where the reliability target is achieved by selecting appropriate design variables. For a given design, to evaluate whether its reliability is satisfactory, reliability analysis needs to be performed to obtain the reliability prediction. If the reliability requirement is not met, RBDO is then carried out to adjust the design variables. This process continues until the reliability and other engineering requirements are satisfied. This process usually involves computer-aided design (CAD), computer-aided engineering (CAE) and optimization.

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There are two kinds of reliability methodology: statistics based and physics based. The statistics-based methods estimate the reliability based on experimental results and/or filed data. The physics-based methods predict the reliability by computational models (limit-state functions) derived from physics. The former methods largely reside in the field of reliability engineering while the latter are often seen in the area of engineering design. In this work, the latter type is used. Since the input variables of the limit-state functions include design variables, such as the major dimensions, it is possible to adjust those design variables during the RBDO process so that an optimal balance between cost and reliability can be reached. Many physics-based reliability methodologies have been developed in recent decades (Huang, Chan, and Lou 2012; Hurtado and Alvarez 2012; Jensen, Kusanovic, and Valdebenito 2012; Kang *et al.* 2012; Sanchez-Silva, Klutke, and Rosowsky 2012). Design optimization with physics-based reliability (Du, Sudjianto, and Huang 2005; Du and Huang 2007; Du 2008; Du, Guo, and Beeram 2008) has been applied in various engineering fields to ensure that a design meets a specified reliability requirement.

The major drawback of traditional physics-based RBDO (Du, Sudjianto, and Huang 2005; Du and Huang 2007; Du 2008; Du, Guo, and Beeram 2008) is that in many applications the reliability is a constant with respect to time. The reason is that the limit-state functions are time independent, *e.g.* a limit-state function involving only static stresses that do not change with time. As a result, there is no way to relate the predicted reliability with time-dependent activities, such as warranty and maintenance. This is the reason that only the initial development cost (not the lifetime cost) usually appears in the objective function in the traditional physics-based RBDO.

In reality, reliability is time dependent. Theoretically, physics-based reliability methodologies are able to produce time-dependent reliability when limit-state functions involve time (Andrieu-Renaud, Sudret, and Lemaire 2004; Singh, Mourelatos, and Li 2010; Zhang and Du 2011; Hu and Du 2012). With the availability of physics-based time-dependent reliability, in the early design stage, the reliability of a product can be predicted after it has operated for a certain period. Various warranty and maintenance activities during the design optimization process can then be taken into account. Studies on physics-based RBDO have been reported. Wang, Du, and Huang (2010) established three physics-based RBDO models to integrate the warranty and maintenance models with the optimization model. Streicher and Rackwitz (2004) proposed a reliability-oriented time-variant structural optimization method for a series system by considering the dependency between failure modes and components with a simple maintenance model. Singh, Mourelatos, and Li (2010) applied the composite limit state method to the lifecycle cost optimization. Most recently, Li, Mourelatos, and Singh (2012) proposed a method for the optimization of a preventive maintenance policy based on lifecycle cost analysis and time-dependent reliability. A review of these physics-based RBDO models (Streicher and Rackwitz 2004; Li, Mourelatos, and Singh 2012) shows that the applications of these models are limited either by their time-dependent reliability analysis methods or by their lifetime optimization models.

Although many lifetime optimization models have been developed based on the statistics-based reliability method (Ahmad and Kamaruddin 2012; Chien, Sheu, and Zhang 2012; Doyen 2012; Herrmann 2012; Pan, Liao, and Xi 2012; Suliman and Jawad 2012; Ye, Shen, and Xie 2012; Zhou *et al.* 2012a; Zhou, Lu, and Xi 2012b), they cannot be applied directly to the design optimization of a product during the parameter design stage, because they are statistics based. The purpose of this work is to develop a lifetime optimization model where the system reliability is predicted by a physics-based reliability method. To ensure high accuracy, a recently developed time-dependent reliability analysis method, which uses a sampling approach to the extreme values of stochastic processes (Hu and Du 2013), is integrated with a lifetime cost model and the RBDO model. By taking account of the costs of product development, reliability, maintenance and warranty, the proposed model can identify optimal design variables so that the product lifetime cost is minimized.

2. Physics-based time-dependent reliability

Physics-based reliability means that the reliability is estimated based on the physical model instead of historical data in early design stages. Suppose that a performance variable G with inputs of random variables $\mathbf{X}$, stochastic processes $\mathbf{Y}(t)$ and time t is given by

$$G = g(\mathbf{X}, \mathbf{Y}(t), t) \tag{1}$$

Define $G = g(\mathbf{X}, \mathbf{Y}(t), t) \leq 0$ as the safe region and $G = g(\mathbf{X}, \mathbf{Y}(t), t) > 0$ as the failure region. For the period $[t_0, t]$, the physics-based time-dependent reliability $R_s(t_0, t)$, which is also called the first passage reliability, is the probability that the response G always remains in the safe region. $R_s(t_0, t)$ is given by

$$R_s(t_0, t) = R_0 \Pr\{G = g(\mathbf{X}, \mathbf{Y}(\tau), \tau) \leq 0, \ \forall\tau \in [t_0, t]\} \tag{2}$$

where R_0 is the initial reliability at time instant t_0.

If the above time-dependent reliability can be predicted from the computational model $g(\cdot)$, the distribution of the product life can also be estimated.

When limit-state functions are time dependent, directly using time-invariant reliability methods may not be applicable. To estimate $R_s(t_0, t)$, time-dependent reliability analysis methods need to be employed. Many time-dependent reliability analysis methods have been proposed, including the most commonly used upcrossing rate methods (Yang and Shinozuka 1971; Bernard and Shipley 1972; Vanmarcke 1975; Ditlevsen 1983; Hu and Du 2012; Hu *et al.* 2013), methods that transform time-dependent problems into time-independent ones (Singh, Mourelatos, and Li 2010; Wang and Wang 2012a, 2012b) and sampling approaches (Singh, Mourelatos, and Nikolaidis 2011). In this work, a newly developed time-dependent reliability analysis method, which employs a sampling approach to the global extreme of a stochastic process and the saddlepoint approximation (Hu and Du 2013), is used. This method is accurate, but only applicable for component reliability. In this work, the method is extended to the system reliability analysis. Thus, the reliability at the product level, or the system reliability, with respect to time, can be predicted computationally.

In the subsequent sections, the optimization model for the lifetime cost is given first, after which the necessary submodels for the optimization model are derived.

3. Lifetime cost optimization with physics-based time-dependent reliability

In this section, the three components of the optimization model, including design variables, objective function and constraint functions, are discussed first. Then, the complete optimization model is presented.

3.1. *Design variables*

In this work, two types of design variable are considered. The first type includes those design variables that appear in traditional engineering optimization problems. Examples include the dimensions of components and system configurations. For example, if a hollow transmission shaft is to be optimized, the design variables could be the inside diameter, the outside diameter, the positions of bearings, and so on. For a transmission system optimization, the design variables could also include the number of gears and number of shafts. With the physics-based reliability method, the link between the design variables and reliability can easily be established using limit-state functions. However, this type of design variable is missing in a statistics-based reliability optimization model because no direct link is available between the design variables and reliability.

The other type of design variables includes those related to product operations, such as the number of preventive maintenances. These variables usually appear in a statistics-based reliability optimization model. The new optimization model in this work can also accommodate this type of design variables.

Some of the design variables may be deterministic and do not change randomly, *e.g.* the number of teeth of a gear, whereas other design variables may be random, *e.g.* the diameter of a shaft is a random design variable because of the random manufacturing imprecision. In this work, for ease of presentation, the mean values of the random design variables are assumed to be determined. However, the proposed model is also applicable to situations where other distribution parameters, such as standard deviations, are involved. In this work, the vector **d** is used to denote all the actual design variables, including both deterministic design variables and the mean vales of random design variables.

3.2. *Objective function*

In the traditional physics-based RBDO model, the objective is often the initial development cost. The present model accounts for the total product lifetime cost, including not only the initial development cost, but also other lifetime associated costs. By minimizing the expected total cost, the optimal design variables **d** can then be found under uncertainties in **X** and $\mathbf{Y}(t)$. The objective function is defined by

$$C_{\text{unit}}(\mathbf{d}) = \frac{C_{\text{life}}(\mathbf{d})}{T_{\text{life}}(\mathbf{d})} \tag{3}$$

in which C_{life} and T_{life} are expected lifetime cost and service life of the product, respectively.

$C_{\text{life}}(\mathbf{d})$ is larger when $T_{\text{life}}(\mathbf{d})$ is longer. This is the reason why the expected unit lifetime cost $C_{\text{unit}}(\mathbf{d})$ is used. Given a specified set of design variables **d**, both $C_{\text{life}}(\mathbf{d})$ and $T_{\text{life}}(\mathbf{d})$ can be derived from the physics-based time-dependent reliability; and so can $C_{\text{unit}}(\mathbf{d})$.

3.3. *Constraints*

Constraints are used to represent requirements or restrictions. A general constraint can be expressed as $g(\mathbf{d}) \leq 0$.

Owing to the involvement of random variables **X** and stochastic processes $\mathbf{Y}(t)$, a constraint may not be satisfied completely, or doing so may be extremely costly. For a constraint $g(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) < 0$ with **X** and $\mathbf{Y}(t)$, it can be satisfied up to a desired or required level. This gives

$$R \geq P \tag{4}$$

where R is the reliability associated with $G = g(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) \leq 0$. It could be the initial reliability at t_0 or time-dependent reliability over $[t_0, t]$. P is the required reliability. The reliability requirement can also be a constraint.

3.4. *Optimization model*

Based on the three components of the optimization model, the new general lifetime cost optimization model is given as follows:

$$\begin{cases} \min\ C_{\text{unit}}(\mathbf{d}) \\ \text{Subject to} \\ R_i \geq P_i, i = 1, 2, \ldots, n_u \\ f_j(\mathbf{d}) \leq 0, j = 1, 2, \ldots, n_d \\ \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U \end{cases} \tag{5}$$

where n_d is the number of deterministic constraints, n_u is the number of probabilistic constraints, $f_j(\mathbf{d})$ is the jth deterministic constraint, and R_i is the reliability of the ith probabilistic constraint.

As the optimization model given in Equation (5) integrates the lifetime cost model with physics-based reliability analysis methods, the model can identify the optimal design variables from the aspect of lifetime cost. Thus, the solutions generated are closer to the true optimal design than those obtained from the traditional physics-based RBDO method.

However, the model given in Equation (5) is just a general model. To apply this model to the optimization of lifetime cost, several submodels are needed. They include component and system reliability models, cost model and lifetime model.

4. Physics-based reliability model

The expected total lifetime cost $C_{\text{life}}(\mathbf{d})$ is a function of product reliability $R_s(t_0, t)$ over the lifetime $[t_0, t]$. $R_s(t_0, t)$ is the product system reliability and depends on component reliability. The physics-based component reliability is discussed next, followed by the physics-based system reliability.

4.1. *Physics-based component reliability analysis*

The method employed is the first order reliability method (FORM), which is commonly used in physics-based reliability analysis and RBDO.

For component i with time-invariant random variables $\mathbf{X}$, assume that the associated limit-state function is $G_i = g_i(\mathbf{d}, \mathbf{X}) = 0$ and that the threshold is e_i. The reliability R_i is computed by

$$R_i = \Phi(\beta_i) = \Phi(\|\mathbf{u}^*\|) \tag{6}$$

Vector $\mathbf{u}^*$ is called the most probable point (MPP), which is obtained by solving the following optimization model:

$$\begin{cases} \min\limits_{\mathbf{u}} \|\mathbf{u}\| \\ \text{subject to } g_i(\mathbf{d},\ T^{-1}(\mathbf{u})) = 0 \end{cases} \tag{7}$$

where $T(\cdot)$ stands for a transformation operator. The transformation is given by

$$\mathbf{U} = T(\mathbf{X}) \tag{8}$$

which transforms the general random variables $\mathbf{X}$ into independent and standard normal variables $\mathbf{U}$. The transformation is given in Choi, Grandhi, and Canfield (2007).

Note that the limit-state function $G_i = g_i(\mathbf{d},\ \mathbf{X}) = 0$ is not time dependent, and neither is the reliability R_i.

When stochastic processes $\mathbf{Y}(t)$ are involved, the associated limit-state function becomes $G_i = g_i(\mathbf{d},\ \mathbf{X}, \mathbf{Y}(t)) = 0$, which is also a stochastic process. For a given period $[t_0,\ t]$, the time-dependent reliability of component i is then given by

$$R_i(t_0, t) = R_{i,0}(\Pr\{G = g_i(\mathbf{d},\ \mathbf{X}, \mathbf{Y}(\tau)) \le 0,\ \forall\tau \in [t_0, t]\}) \tag{9}$$

To estimate $R_i(t_0, t)$, in this work, a newly developed sampling approach is employed (Hu and Du 2013). Using this method, the time-dependent problem is transformed to a time-independent problem, and $R_i(t_0, t)$ is calculated by

$$R_i(t_0, t) = R_{i,0}(\Pr\{G = g_i(\mathbf{d},\ \mathbf{X}, \mathbf{Y}_{\text{ext}}) \le 0\}) \tag{10}$$

in which $\mathbf{Y}_{\text{ext}}$ is the extreme value of $\mathbf{Y}(t)$ over $[t_0, t]$.

After the transformation, a time-independent reliability method such as FORM can be applied to estimate $R_i(t_0, t)$.

To obtain the distributions of $\mathbf{Y}_{\text{ext}}$, The Monte Carlo simulation (MCS) method is employed. The simulation is performed only on stochastic process $\mathbf{Y}(t)$, and it does not call the limit-state function. In engineering applications, calling a limit-state function may be time consuming because it may involve expensive simulations such as a finite element analysis. Without evaluating a limit-state function, the sampling method for $\mathbf{Y}_{\text{ext}}$ is efficient. More details about the sampling method can be found in Hu and Du (2013).

It should be noted that the requirements of the sampling method are that there is only one stochastic process in $\mathbf{Y}(t)$ and that the process is either a general strength variable or a general stress variable (Hu and Du 2013). However, these restrictions can be removed if other methods are used.

4.2. *Physics-based system reliability analysis method*

Typical systems can be classified into three categories: parallel systems, series systems and combined systems. Herein, the physics-based system reliability analysis for series systems is focused on as the parallel system can be transformed into series system. Using the bounding formulae given by Ditlevsen (1979), the probability of failure of a series system can be approximated with the following bounds:

$$p_{f,\min} \leq 1 - R_s(t_0, t) \leq p_{f,\max} \tag{11}$$

where

$$p_{f,\min} = [1 - R_1(t_0, t)] + \sum_{i=2}^{n} \max\{[1 - R_i(t_0,\ t)] - \sum_{j=1}^{i-1} P_{ij}(t_0, t), 0\} \tag{12}$$

$$p_{f,\max} = \sum_{i=1}^{n} [1 - R_i(t_0, t)] - \sum_{i=2}^{n} \max_{j<i} P_{ij}(t_0, t) \tag{13}$$

where $R_1(t_0, t)$ is the reliability of the component whose probability of failure is the largest. $P_{ij}(t_0, t)$ is the probability that both components i and j fail over the time interval $[t_0, t]$.

From Equations (12) and (13), it can be seen that $R_i(t_0, t)$ and $P_{ij}(t_0, t)$ are the bases for the physics-based system reliability analysis. Suppose $G_i = g_i(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) = 0$ and $G_j = g_j(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) = 0$ are limit-state functions for components i and j, $P_{ij}(t_0, t)$ is then given by

$$\begin{aligned} P_{ij}(t_0, t) = \Pr\{G_i = g_i(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\chi)) > 0 \cap \\ G_j = g_j(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) > 0, \exists \chi \text{ and } \tau \in [t_0, t]\} \end{aligned} \tag{14}$$

Song and Der Kiureghian (2006) proposed a method to solve Equation (14) using the Rice formula (Rice 1944, 1945). Their method, however, only focuses on problems with multiple stationary Gaussian stochastic processes. The new method is more general because it can be used for non-stationary processes. By applying the sampling-based method discussed in Hu and Du (2013), the new method transforms limit-state functions $G_i = g_i(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) = 0$ and $G_j = g_j(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) = 0$ into

$$\Pr\{G_i = g_i(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) > 0, \exists \tau \in [t_0, t]\} = \Pr\{G_i = g_i(\mathbf{d}, \mathbf{X}, Y_{\text{ext, i}}) > 0\} \tag{15}$$

and

$$\Pr\{G_j = g_j(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) > 0, \exists \tau \in [t_0, t]\} = \Pr\{G_j = g_j(\mathbf{d}, \mathbf{X}, Y_{\text{ext, j}}) > 0\} \tag{16}$$

The time-invariant random variables are transformed into standard normal space and MPPs are searched for G_i and G_j using FORM. Once the MPPs are available, Equations (15) and (16) are further transformed into

$$\Pr\{G_i = g_i(\mathbf{d}, \mathbf{X}, \mathbf{Y}_{\text{ext}}) > 0\} \approx \Pr\{L_i = \boldsymbol{\alpha}_i \mathrm{U}_i > \beta_i\} \tag{17}$$

and

$$\Pr\{G_j = g_j(\mathbf{d}, \mathbf{X}, \mathbf{Y}_{\text{ext}}) > 0\} \approx \Pr\{L_j = \boldsymbol{\alpha}_j \mathbf{U}_j > \beta_j\} \tag{18}$$

in which $\mathbf{U}_j$ are transformed variables from $\mathbf{X}$ and $\mathbf{Y}_{\text{ext}}$ in the standard normal space, and L is the linearized state response G in the standard normal space.

$$\boldsymbol{\alpha}_i = \frac{\mathbf{u}_i^*}{\|\mathbf{u}_i^*\|} \text{ and } \boldsymbol{\alpha}_j = \frac{\mathbf{u}_j^*}{\left\|\mathbf{u}_j^*\right\|} \tag{19}$$

$$\beta_i = \|\mathbf{u}_i^*\| \text{ and } \beta_j = \left\|\mathbf{u}_j^*\right\| \tag{20}$$

where $\mathbf{u}_i^*$ and $\mathbf{u}_j^*$ are obtained using Equation (7).

With Equations (14)–(20), the joint probability of failure $P_{ij}(0, t)$ of components i and j is then computed by

$$P_{ij}(t_0, t) = \int_{\beta_i}^{\infty} \int_{\beta_j}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{L_i^2 + L_j^2 - 2\rho L_i L_j}{2(1-\rho^2)}\right\} dL_j dL_i \tag{21}$$

in which

$$\rho = \boldsymbol{\alpha}_i \boldsymbol{\alpha}_j^T \tag{22}$$

After $R_i(t_0, t)$ and $P_{ij}(t_0, t)$ have been solved, the physics-based system reliability can be estimated using Equations (11)–(13).

5. Lifetime cost model

With the availability of the system reliability function of time, it is possible to estimate the product lifetime cost in the parameter design stage. The cost will be a direct function of design variables, and then the cost model can be used for the objective function to be minimized.

The lifetime cost $C_{\text{life}}(\mathbf{d})$ consists of two parts: the initial development cost C_{I} and maintenance cost C_{M}. Then,

$$C_{\text{life}}(\mathbf{d}) = C_{\text{I}} + C_{\text{M}} \tag{23}$$

Next, the equations for C_{I} and C_{M} are derived based on the system reliability function $R_s(t_0, t)$.

5.1. *Initial cost C_{I}*

The initial cost includes the design, development and production costs (Wang, Du, and Huang 2010). It depends on many factors. For example, better materials and larger dimensions usually imply a higher initial cost. Higher reliability may be also associated with a higher initial investment. A cost model normally relies on specific products, and a universal cost model may not be available for all products. This is the reason why many cost models exist (Krishnaswami and Mayne 1994;

Huang, Liu, and Murthy 2007; Kalowekamo and Baker 2009). Take the model proposed by Mettas (2000) as an example. The model is given by

$$C_{\mathrm{I}} = A_1 + B_1 \exp(qR_0) \tag{24}$$

The model indicates that the initial cost increases exponentially with respect to the initial reliability. The cost goes up slowly when the initial reliability is low and rises rapidly when the initial reliability is high. The parameters in Equation (24) are determined according to either the initial cost data of similar products or empirical experience. It is a simple model that directly relates the initial cost to the initial system reliability. As mentioned previously, a more advanced initial cost model may also be used.

5.2. *Maintenance cost C_{M}*

The maintenance cost C_{M} is directly related to the system reliability. To derive equations for the maintenance cost C_{M}, the general maintenance policy is discussed first.

5.2.1. *Maintenance policy*

For products with a warranty, as shown in Figure 1, the product lifetime is divided into the warranty period and post-warranty period. In this work, only the one-dimensional warranty policy is considered, which covers a specified period and does not cover the amount of use.

Typically, the type of maintenance to be implemented is determined by the type of failure. Failure may not always be catastrophic; sometimes, its effects may be minor or moderate. According to the economic loss, failures may be classified into two categories:

- Type I failure: a minor or moderate failure that does not affect the operation of the system
- Type II failure: a catastrophic failure that can be removed only by replacement over the warranty period or by corrective maintenance over the post-warranty period.

Let the probabilities of type I and II failures be p_1 and p_2, respectively. They satisfy

$$p_1 + p_2 = 1 \tag{25}$$

The summation is one, which indicates that failure events can be either a type I failure or a type II failure. p_1 and p_2 may be determined by field data of existing similar products, assumptions or other means. In this work, p_1 and p_2 are assumed to be known.

The maintenance activities are defined as follows:

- *Minimal maintenance*: maintenance activities performed to the system whenever a type I failure occurs. Such activities do not affect the failure rate of the system.
- *Corrective maintenance*: maintenance activities performed to the system to remove type II failures over the post-warranty period.
- *Preventive maintenance*: maintenance activities conducted over the post-warranty period to guarantee the reliability of the system. Preventive maintenance is performed whenever the

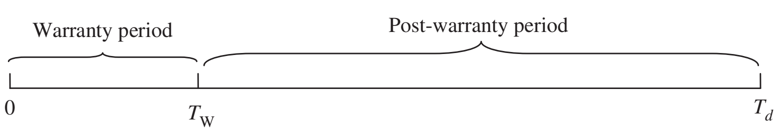

Figure 1. Warranty period and post-warranty period of a system.

system operation time reaches the planned period T since the last preventive maintenance or corrective maintenance.
- *Replacement*: activities performed to the system to recover the system from type II failures over the warranty period. After the replacement, the system starts with a new warranty period.

A corrective maintenance may not completely recover the system from type II failures. Herein, a positive coefficient $\gamma_1 \leq 1$ is used to quantify the capacity of corrective maintenances (Wang, Du, and Huang 2010). Define $R_{\mathrm{I},i}$ as the immediate system reliability after the ith maintenance. If the ith maintenance is corrective,

$$R_{\mathrm{I},i} = \gamma_1 R_0 \tag{26}$$

Similarly, another coefficient $0 \leq \gamma_2 \leq 1$ is used to describe the capacity of preventive maintenance. $\gamma_2 = 1$ means that the product will be as good as new after preventive maintenance, while $\gamma_2 = 0$ indicates that the product will be as bad as old after preventive maintenance; otherwise, the product will be between the old and new states.

Let $R_{s,i-1}$ be the system reliability right before the ith maintenance. If the ith maintenance is preventive, after it, the system reliability $R_{\mathrm{I},i}$ is then

$$R_{\mathrm{I},i} = R_{s,i-1} + \gamma_2 (R_0 - R_{s,i-1}) \tag{27}$$

Equation (27) indicates that when $\gamma_2 = 0$, $R_{\mathrm{I},i} = R_{s,i-1}$ and that when $\gamma_2 = 1$, $R_{\mathrm{I},i} = R_0$.

With Equations (26) and (27), $R_{\mathrm{I},i}$ after the ith maintenance is computed as follows:

$$R_{\mathrm{I},i} = \begin{cases} \gamma_1 R_0, & \text{if Type II failure occurs between } (i-1)-\text{th and} \\ & i\text{-th maintenances} \\ R_{s,\,i-1} + \gamma_2 (R_0 - R_{s,\,i-1}), & \text{otherwise} \end{cases} \tag{28}$$

Based on the above maintenance polices, the expected total maintenance cost C_{M} is given by

$$C_{\mathrm{M}} = C_{\mathrm{W}} + C_{\mathrm{P}} \tag{29}$$

in which C_{W} and C_{P} are the expected maintenance cost over the warranty period and post-warranty period, respectively.

5.2.2. *Expected maintenance cost over the warranty period* C_{W}

The maintenance cost over the warranty period, C_{W}, is given by

$$C_{\mathrm{W}} = C_{\mathrm{WI}} + C_{\mathrm{WII}} \tag{30}$$

C_{WI} is given by

$$C_{\mathrm{WI}} = C_{\min} N_{\mathrm{I}} \tag{31}$$

where $C_{\min}$ is the expected minimal maintenance cost per time and N_{I} is number of type I failures.

Recall that a type II failure is removed by replacement in the warranty period. C_{WII} is therefore given by

$$C_{\text{WII}} = N_{\text{r}} C_{\text{I}} \tag{32}$$

Since the system starts with a new warranty period after each replacement, the probability that a replacement is the last one over the warranty period is given by

$$p_r = p_1 + p_2 R_s(0,\ T_W) \tag{33}$$

The expected number of replacements N_{r} follows a geometric distribution (Bain and Engelhardt 1992) with parameter p_r. N_{r} is given by

$$N_{\text{r}} = 1/p_r = 1/(p_1 + p_2 R_s(0,\ T_W)) \tag{34}$$

For each replacement before the last replacement, the expected operation time, T_{r}, between two replacements is computed by

$$T_{\text{r}} = p_2 \int_0^{T_W} R_s(0,\ t)dt - p_2 T_W R_s(0,\ T_W) \tag{35}$$

With the lifetime distribution from the physics-based time-dependent reliability analysis, the expected number of type I failures, N_{WI}, is then computed by (Wang, Du, and Huang 2010)

$$N_{\text{WI}} = \int_0^{T_r} \lambda(t)dt = -\ln[R_s(t_0, T_{\text{r}})] \tag{36}$$

where $t_0 = 0$ and $\lambda(t)$ is the failure rate at time instant t.

For the last replacement, the expected number of type I failures, $N_{\text{WI-last}}$, is given by

$$N_{\text{WI-last}} = -\ln[R_s(0,\ T_W)] \tag{37}$$

With Equations (31), (34), (36) and (37), N_{I} is given by

$$\begin{aligned} N_{\text{I}} =& (N_{\text{r}} - 1)N_{\text{WI}} + N_{\text{WI-last}} \\ =& -[1/(p_1 + p_2 R_s(0,\ T_W)) - 1]\ln[R_s(t_0,\ T_r)] - \ln[R_s(0,\ T_W)] \end{aligned} \tag{38}$$

Plugging Equations (31) and (38) into Equation (30), the expected maintenance cost over the warranty period is obtained as follows:

$$\begin{aligned} C_{\text{W}} =& [1/(p_1 + p_2 R_s(0,\ T_W)) - 1]\{C_{\min}\ln[1/R_s(0,\ T_{\text{r}})] + C_{\text{I}}\} \\ & - C_{\min}\ln[R_s(0,\ T_W)] \end{aligned} \tag{39}$$

5.2.3. *Expected maintenance cost over the post-warranty period C_{P}*

Suppose that during the post-warranty period the system is maintained N times, including the preventive maintenance with an expected cost C_{p1} and corrective maintenance with an expected cost C_{p2}. After the N maintenances, the system will be replaced with a new one. During the post-warranty period $[T_W, T_d]$, a preventive maintenance is performed whenever the operation time of the system reaches $T = (T_d - T_W)/N$ since the last preventive or corrective maintenance. When a type II failure occurs before the next preventive maintenance, a corrective maintenance is performed immediately to correct the failure. Defining the N maintenances as N states of the system, the maintenance model over the post-warranty period is depicted as in Figure 2.

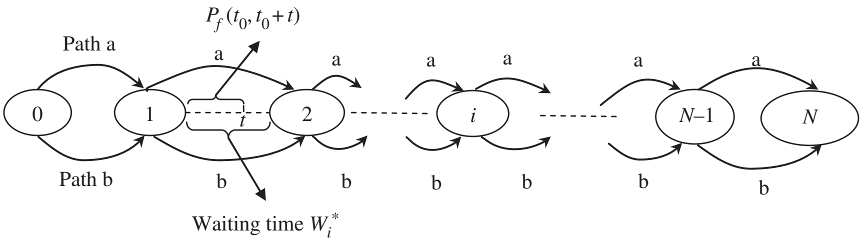

Figure 2. Maintenance policy over the post-warranty period.

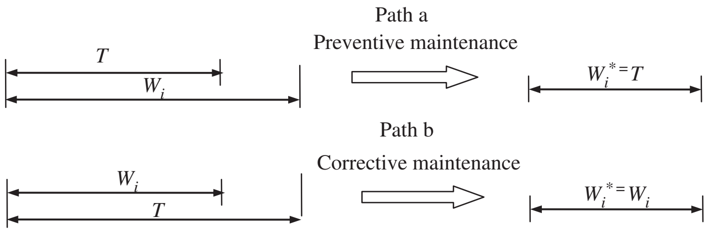

Figure 3. Two paths of maintenance activities.

As indicated in Figure 2, there are two paths for the state transition of the system. The system goes from one maintenance state to the next one, either by reaching the planned preventive maintenance time T or by correcting a type II failure. The two paths are demonstrated in Figure 3. Define W_i^* as the waiting time between two successive maintenances and W_i as the waiting time of a type II failure since the last maintenance.

For Path *a*, the system is preventively maintained because the waiting time between two successive type II failures W_i is longer than the planned preventive maintenance period T. For Path *b*, the system is correctively maintained as the type II failure occurs within the planned preventive maintenance period T. The waiting time W_i^* between two successive maintenance states is therefore given by

$$W_i^* = \begin{cases} W_i, & \text{if } W_i \leq T \\ T, & \text{if } W_i > T \end{cases} \tag{40}$$

The waiting time W_i^* is a random variable associated with the system reliability. Sheu *et al.* (2001) derived the expressions for W_i^* and the associated maintenance cost. Their derivations, however, assume that the failures are independent of each other and are based on the statistics-based reliability method. Besides, they have not considered the effect of warranty on the maintenance cost. In this section, the physics-based reliability analysis method is applied to derive the preventive maintenance cost. As there are two paths for the system to go from one maintenance state to the next, the expected maintenance cost $C_{\text{pu},i}$ for transition i (*i.e.* from maintenance $i-1$ to maintenance i) is given by

$$C_{\text{pu},i} = P_{\text{a},i}C_{\text{pa},i} + P_{\text{b},i}C_{\text{pb},i} \tag{41}$$

in which $P_{\text{a},i}$ (*i.e.* $\Pr\{W_i > T\}$) and $P_{\text{b},i}$ (*i.e.* $\Pr\{W_i \leq T\}$) are the probabilities that the system has transition i by Paths *a* and *b*, respectively, and $C_{\text{pa},i}$ and $C_{\text{pb},i}$ are corresponding expected maintenance costs, respectively.

$P_{\text{a},i}$ and $P_{\text{b},i}$ are given by

$$P_{\text{a},i} = 1 - p_2(1 - R_{s,i-1}) = p_1 + p_2 R_{s,i-1} \tag{42}$$

and

$$P_{\mathrm{b},i} = p_2(1 - R_{s,i-1}) \tag{43}$$

where $R_{s,i-1} = R_{I,\ i-1}R_s(0,\ T)/R_0$.

Since there are two possible paths between two successive maintenances, there are 2^{i-1} possible paths for the system to reach its ith maintenance from its initial condition. For the jth path for $P_{\mathrm{a},i}$ and $P_{\mathrm{b},i}$,

$$P_{\mathrm{a},i,j} = 1 - p_2(1 - R_{\mathrm{s},i-1,j}) = p_1 + p_2R_{\mathrm{s},i-1,j}, j = 1, 2, \ldots, 2^{i-1} \tag{44}$$

and

$$P_{\mathrm{b},i,j} = p_2(1 - R_{\mathrm{s},i-1,j}),\ j = 1, 2, \ldots, 2^{i-1} \tag{45}$$

in which

$$R_{\mathrm{s},i-1,j} = R_{\mathrm{I},\ i-1,j}R_s(0,\ T)/R_0,\ j = 1, 2, \ldots, 2^{i-1} \tag{46}$$

The probability that $R_{\mathrm{I},i}$ is equal to $R_{\mathrm{I},i,j}$ is given by

$$P_{i+1,\ j} = \Pr\{R_{\mathrm{I},\ i} = R_{\mathrm{I},\ i,j},\ j = 1, 2, \cdots, 2^i\} = \begin{cases} P_{\mathrm{a},i,\ j+1}P_{i,\ (j+1)/2}, & \text{if } \ j \text{ is odd} \\ P_{\mathrm{b},i,\ j/2}P_{i,\ j/2}, & \text{if } \ j \text{ is even} \end{cases} \tag{47}$$

where $P_{1,\ 1} = 1$.

With Equations (44)–(46), Equation (41) becomes

$$C_{\mathrm{pu},i,j} = P_{a,i,j}C_{\mathrm{pa},i,j} + P_{b,i,j}C_{\mathrm{pb},i,j} \tag{48}$$

and

$$C_{\mathrm{p},i} = \sum_{j=1}^{2^{i-1}}(P_{i,j}C_{\mathrm{pu},i,j}) \tag{49}$$

in which $C_{\mathrm{p},i}$ stands for the expected maintenance cost for transition i, and $C_{\mathrm{pa},i,j}$ and $C_{\mathrm{pb},i,j}$ are expected maintenance costs of transition i for Paths a and b given that $R_{I,i-1}$ equals $R_{I,i-1,j}$, respectively.

For Path a, the preventive maintenance is performed before a type II failure occurs. $C_{\mathrm{pa},i,j}$ is therefore given by

$$C_{\mathrm{pa},i,j} = C_{p1} + \ln\{1/R_s(0, T)\}C_{\min}, j = 1, 2, \ldots, 2^{i-1} \tag{50}$$

For Path b, the corrective maintenance is carried out before the operation time reaches T, and C_{pb} is derived to be

$$C_{\mathrm{pb},i,j} = p_2\frac{R_{I,i-1,j}}{R_0}\left\{C_{p2}[1 - R_s(0, T)] + C_{\min}\{R_s(0, T)\ln[R_s(0, T)] - R_s(0, T) + 1\}\right\} \tag{51}$$

Plugging Equations (50) and (51) into Equation (49) and after simplification, $C_{\mathrm{pu},\ i,j}$ is

$$\begin{aligned} C_{\mathrm{pu},i,j} = &[p_1 + p_2R_{s,i-1,j}]C_{p1} + p_1C_{\min}\ln[1/R_s(0, T)] \\ &+ p_2^2\frac{R_{I,i-1,j}}{R_0}(1 - R_{s,i-1,j})(C_{\min} + C_{p2})[1 - R_s(0, T)] \\ &+ p_2C_{\min}R_{s,i-1,j}\ln[1/R_s(0, T)][p_1 + p_2R_{s,i-1,j}] \end{aligned} \tag{52}$$

For the last cycle, there is no preventive maintenance, and the system is replaced with a new one. The expected maintenance cost $C_{\text{pu},\,N,j}$ is given by

$$\begin{aligned} C_{\text{pu},N,j} =& C_{\text{I}} + p_1 C_{\min} \ln[1/R_s(0,T)] \\ &+ p_2^2 C_{\min} \frac{R_{I,i-1,j}}{R_0}(1 - R_{s,i-1,j})[1 - R_s(0,T)] \\ &+ p_2 C_{\min} R_{s,i-1,j} \ln[1/R_s(0,T)](p_1 + p_2 R_{s,i-1,j}) \end{aligned} \tag{53}$$

and

$$C_{\text{p},N} = \sum_{j=1}^{2^{N-1}} \{P_{N,j} C_{\text{pu},N,j}\} \tag{54}$$

Since the system is maintained N times, C_{P} is given by

$$C_{\text{P}} = \sum_{i=1}^{N} C_{\text{p},i} = \sum_{i=1}^{N} \sum_{j=1}^{2^{i-1}} (P_{i,j} C_{\text{pu},i,j}) \tag{55}$$

Equations (50)–(55) indicate that the initial reliabilities $R_{I,i}, i = 1, 2, \ldots, N$ need to be solved to compute the expected maintenance cost over the post-warranty period. To solve the initial reliabilities, the procedure given in Figure 4 is followed iteratively.

The iterative process starts from $R_{s,0}$, which is given by

$$R_{s,0} = R_{I,0} R_s(0,T)/R_0 \tag{56}$$

Since the last replacement survives over the warranty period $[0, T_W]$, $R_{I,0}$ is given as follows:

$$R_{I,0} = R_s(0, T_W) \tag{57}$$

With Equations (56) and (57), the initial reliabilities and associated probabilities can be solved using Equations (44)–(46) and (28) iteratively. So far, all the equations needed for the maintenance costs over both warranty period and post-warranty period have been obtained.

5.3. *Lifecycle cost C_{life}*

Based on the discussion on various costs, the expected lifecycle cost can now be obtained. Substituting Equations (24), (29), (39) and (55) into Equation (23), the expected lifecycle cost C_{life}

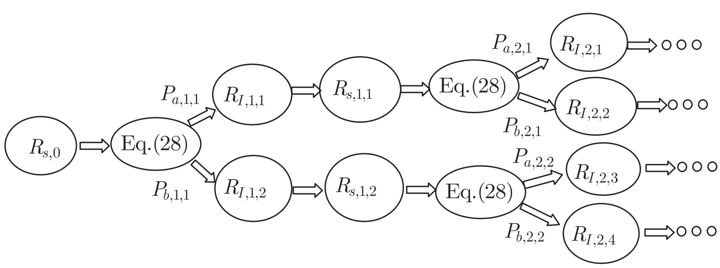

Figure 4. Iteration procedure for solving initial reliabilities.

turns out to be

$$\begin{aligned} &C_{\text{life}}(\mathbf{d}) \\ &= C_{\text{I}} + [1/(p_1 + p_2 R_s(0, T_W)) - 1]\{C_{\min} \ln[1/R_s(t_0, T_{\text{r}})] + C_{\text{I}}\} \\ &\quad + C_{\min} \ln[1/R_s(0, T_W)] + \sum_{i=1}^{N} \sum_{j=1}^{2^{i-1}} \{P_{i,j} C_{\text{pu},i,j}\} \end{aligned} \tag{58}$$

5.4. *Expected service life T_{life}*

In order to apply the optimization model given in Equation (5) for the life cost optimization, the expected service life T_{life} is also needed. The expected service life consists of the warranty period and the post-warranty period. The latter is determined by the expected waiting time $T_{p,i}$ between two successive maintenances. $T_{p,i}$ between the $(i-1)$th and ith transitions is given by

$$T_{p,i} = \sum_{j=1}^{2^{i-1}} (P_{i,j} T_{\text{pu},i,j}) \tag{59}$$

where $T_{\text{pu},i,j}$ is the expected waiting time for transition i given that $R_{I,i-1}$ equals $R_{I,i-1,j}$, and is computed by

$$T_{\text{pu},i,j} = TP_{a,i,j} + P_{b,i,j} \left[p_2 \frac{R_{I,i-1,j}}{R_0} \int_0^T R_s(0,t)dt - Tp_2 R_{s,i-1,j} \right] \tag{60}$$

Substituting Equation (60) into (59),

$$T_{p,i} = \sum_{j=1}^{2^{i-1}} \left\{ P_{i,j} \left\{ TP_{a,i,j} + P_{b,i,j} \left[p_2 \frac{R_{I,i-1,j}}{R_0} \int_0^T R_s(0,t)dt - Tp_2 R_{s,i-1,j} \right] \right\} \right. \tag{61}$$

Therefore, the expected service life is given by

$$\begin{aligned} T_{\text{life}} =& [1/(p_1 + p_2 R_s(0,\ T_W)) - 1][p_2 \int_0^{T_W} R_s(0,t)dt - p_2 T_W R_s(0,T_W)] + T_W \\ &+ \sum_{i=1}^{N} \sum_{j=1}^{2^{i-1}} \left\{ P_{i,j}\ TP_{a,i,j} + P_{b,i,j} \left[p_2 \frac{R_{I,i-1,j}}{R_0} \int_0^T R_s(0,t)dt - Tp_2 R_{s,i-1,j} \right] \right\} \Bigg\} \end{aligned} \tag{62}$$

Plugging Equations (58) and (62) into Equation (3) yields

$$C_{\text{unit}} = \frac{\left\{ \begin{matrix} C_{\text{I}} + [1/(p_1 + p_2 R_s(0, T_W)) - 1]\{C_{\min} \ln[1/R_s(t_0, T_{\text{r}})] + C_{\text{I}}\} \\ + C_{\min} \ln[1/R_s(0, T_W)] + \sum_{i=1}^{N} \sum_{j=1}^{2^{i-1}} (P_{i,j} C_{pu,i,j}) \end{matrix} \right\}}{\begin{matrix} [1/(p_1 + p_2 R_s(0, T_W)) - 1][p_2 \int_0^{T_W} R_s(0,t)dt - p_2 T_W R_s(0,T_W)] + T_W \\ + \sum_{i=1}^{N} \sum_{j=1}^{2^{i-1}} \left\{ P_{i,j} \left\{ TP_{a,i,j} + P_{b,i,j} \left[p_2 \frac{R_{I,i-1,j}}{R_0} \int_0^T R_s(0,t)dt - Tp_2 R_{s,i-1,j} \right] \right\} \right. \end{matrix}} \tag{63}$$

where

$$T = (T_d - T_W)/N \tag{64}$$

With the formula given in Equation (63) and the connection between design variables and time-dependent reliability, the lifetime cost design optimization model is then complete, as shown in Equation (5).

6. Numerical procedure

Here, the numerical procedure is summarized for the proposed lifetime cost optimization. Figure 5 shows the flowchart.

In the proposed model, there are two main modules: the sample generation module and the optimization module. The purpose of the sampling generating module is to estimate the global extreme value distributions of stochastic processes over different periods. The generated extreme distributions from the sampling generating module are then used as the input to the optimization module. The optimization module consists of three different models, including the design optimization model, reliability analysis models and the lifecycle cost model. These three models are coupled together by the design variables, reliabilities and cost variables.

By solving the proposed model, it can obtain not only the optimal design variables, but also the associated optimal warranty period and preventive maintenance period.

7. Examples

Two design optimization examples with time-dependent uncertainties are used to demonstrate the proposed method. They are a beam under stochastic loading and a Daniels structural system. The first example has one component and the second one is a parallel system with two components.

7.1. *Example 1: Design optimization of a beam under stochastic loading*

Figure 6 shows a simply supported beam under stochastic loading. The cross-section of the beam is rectangular, and the width a and height b are to be optimized to minimize the expected unit lifetime cost. The length of the beam is L, and a stochastic force $F(t)$ is applied at the mid-span.

The major failure mode of the beam is the breakage due to excessive normal stress. The limit-state function of the beam is therefore given by

$$g(\mathbf{X},\ \mathbf{Y}(t)) = (F(t)L/4 + \rho_{st}abL^2/8) - ab^2\sigma_u/4 = 0 \tag{65}$$

in which $\mathbf{X} = [a,\ b,\ \sigma_u]$, $\mathbf{Y}(t) = [F(t)]$. The uncertain and deterministic variables are given in Table 1.

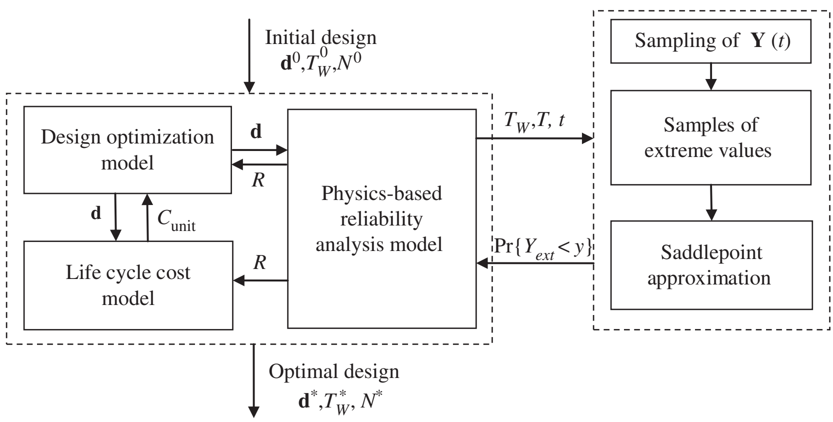

Figure 5. Flowchart of lifecycle cost-optimization model.

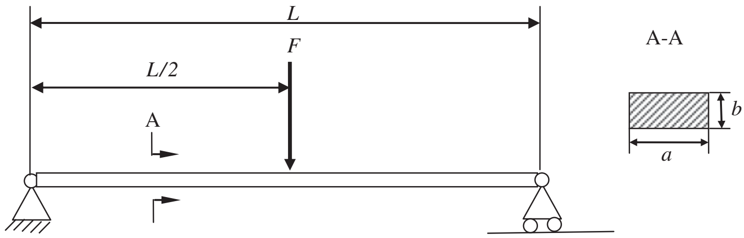

Figure 6. A simply supported beam under dead load and stochastic loading.

Table 1. Deterministic parameters and random variables.

Variable	Mean	Standard deviation	Distribution	Autocorrelation
a	Design variable	0.01 m	Normal	N/A
b	Design variable	5×10^{-3} m	Normal	N/A
σ_u	2.4×10^8 Pa	2.4×10^7 Pa	Normal	N/A
$F(t)$	4500 N	900 N	Gaussian	Equation (66)
L	20	0	Deterministic	N/A
ρ_{st}	7.85×10^4	0	Deterministic	N/A

Table 2. Optimal results with different p_1.

p_1	Optimized results				
	μ_a	μ_b	T_W	N	C_{unit}
0.2	0.6408 m	0.1059 m	118.49 months	12.68	$2.4389 × 10^5
0.4	0.4804 m	0.1087 m	47.16 months	12.00	$2.4372 × 10^5
0.6	0.4259 m	0.1099 m	60.26 months	9.00	$2.4265 × 10^5
0.8	0.4281 m	0.1098 m	59.94 months	6.21	$2.4184 × 10^5

The auto-correlation function of the stochastic process $F(t)$ is given by

$$\rho_F(t_1, t_2) = \exp(-4(t_2 - t_1)^2) \tag{66}$$

where t_1 and t_2 are time in months. The design life of the beam is 30 years or 360 months. The initial reliability of the beam should not be less than 0.999. The design optimization model for the beam problem is then given by

$$\begin{cases} \min \ C_{\text{unit}}(\mathbf{d}) \\ \text{Subject to} \\ \Pr\{g(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t_0)) > 0\} \le 0.001 \\ C_{\text{I}} = 5 \times 10^5 + 36 \exp\{14 R_s(\mathbf{d}, t_0)\} \\ \mathbf{d}^L < \mathbf{d} < \mathbf{d}^U \end{cases} \tag{67}$$

where $C_{\text{unit}}(\mathbf{d})$ is given in Equation (63), $\mathbf{d} = [\mu_a, \mu_b, N, T_W]$, $T = (360 - T_W)/N$, $\mathbf{d}^L = [0.41, 0.10, 1, 24]$, $\mathbf{d}^U = [0.65, 0.15, 20, 120]$, $C_{p1} = \$\ 1 \times 10^5$, $C_{p2} = \$\ 1.8 \times 10^6$, $C_{\min} = \$\ 6 \times 10^4$, $\gamma_1 = 0.9$, $\gamma_2 = 0.8$, $p_1 = 0.8$, and $p_2 = 0.2$.

The initial design points are $\mathbf{d}^0 = [\mu_a^0, \mu_b^0, N^0, T_W^0] = [0.42\ \text{m}, 0.11\ \text{m}, 5\ \text{times}, 60\ \text{months}]$. The optimized design variables are given in the row where $p_1 = 0.8$ in Table 2.

To study the effect of failure types on the optimization results, the beam was optimized with different values of p_1. The results are also given in Table 2.

Table 3. Optimal results with different γ_2.

γ_2	Optimized results				
	μ_a	μ_b	T_W	N	C_{unit}
0.4	0.4287 m	0.1098 m	60.41 months	6.27	\$2.4195 × 10^5
0.6	0.4286 m	0.1098 m	59.91 months	6.26	\$2.4189 × 10^5
0.8	0.4281 m	0.1098 m	59.94 months	6.21	\$2.4184 × 10^5

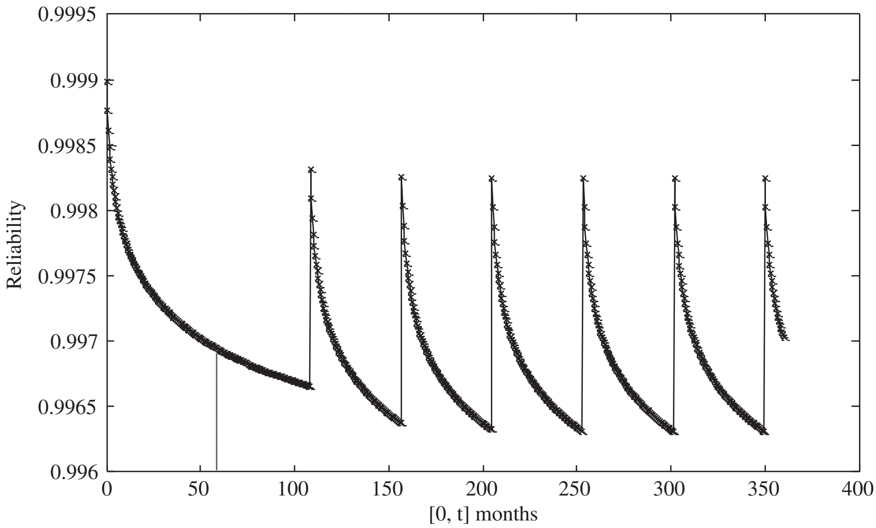

Figure 7. Time-dependent reliability.

The results indicate that the number of preventive maintenances and the expected unit lifetime cost decrease as p_1 increases. This means that a higher probability of type I failures requires fewer preventive maintenances. It is noted the optimized numbers of maintenances are not integers; for example, when $p_1 = 0.2$, the number of maintenances is 12.68, and 12 maintenances are expected during the post-warranty period. Since the warranty period is 118.49 months, the planned maintenance period is $T = (360 - 118.49)/12.68$ months, and the system will continue to operate for $0.68T = 12.95$ months after the maintenance and before it is replaced.

The influence of the capacity of preventive maintenance was also investigated using different values of γ_2. The optimal results are given in Table 3. The result demonstrates that the required number of maintenances decreases when the capacity of maintenance increases.

Figure 7 shows the time-dependent reliability for the case where $p_1 = 0.8$ and $\gamma_2 = 0.8$ at the optimal designs. As shown in Figure 7, the time-dependent reliability is disturbed by the preventive maintenance. The probability of failure increases with time during the warranty period. For the post-warranty period, six preventive maintenances are scheduled to guarantee the probability of failure below a certain level and thus minimize the expect unit lifetime cost.

7.2. *Example 2: Design optimization of a Daniels system*

The second example is the optimization design of a Daniels system as shown in Figure 8, which indicates that the system is in parallel. This example is adapted from McDonald and Mahadevan (2008) with the random loading changed to a stochastic process.

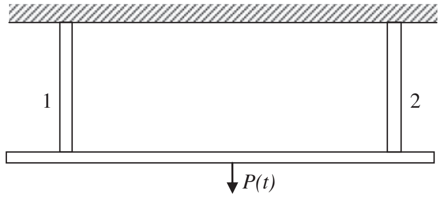

Figure 8. A two-bar system.

Table 4. Random variables.

Variable	Mean	Standard deviation	Distribution	Autocorrelation
S_1	Design variable	0.1 in^2	Normal	N/A
S_2	Design variable	0.1 in^2	Normal	N/A
$\sigma_{b,1}$	36 ksi	3 ksi	Normal	N/A
$\sigma_{b,2}$	36 ksi	3 ksi	Normal	N/A
$P(t)$	90 kips	9 kips	Gaussian	Equation (69)

Each of the two bars is assumed to resist a load of $P(t)/2$ until both of the two bars are yielded. The limit-state functions for the components are given by

$$g_i(\mathbf{X},\ \mathbf{Y}(t)) = S_i\sigma_{b,i} - P(t)/2 = 0,\ i = 1,\ 2 \tag{68}$$

The design life of the beam is 10 years or 120 months. The cross-sections of the two bars are to be optimized. At the same time, the associated optimal warranty period and preventive maintenance policy are expected to be identified. The initial probability of failure of the system is required to be less than 1×10^{-3}.

The cross-section and yield stress of the two bars are independently distributed. There are therefore four design variables: the cross-sections of the two bars, the warranty period, and the preventive maintenance period over the post-warranty period. The deterministic parameters and random variables are given in Table 4.

The auto-correlation function of the stochastic process $P(t)$ is given by

$$\rho_P(t_1,\ t_2) = \exp\left[-\left(\frac{(t_2 - t_1)}{3}\right)^2\right] \tag{69}$$

where t_1 and t_2 are in months. Based on the above information, the optimization model is given as follows:

$$\begin{cases} \min\ C_{\text{unit}}(\mathbf{d}) \\ \text{Subject to} \\ C_{\text{I}} = 1000 + 18\exp\{10R_s(\mathbf{d},\ t_0)\} \\ R_0(\mathbf{d}) \geq 0.999 \\ \mathbf{d}^L < \mathbf{d} < \mathbf{d}^U \end{cases} \tag{70}$$

where $\mathbf{d} = [\mu_{S_1},\ \ \mu_{S_2},\ \ N,\ \ T_W]$, $T = (120 - T_W)/N$, $\mathbf{d}^L = [1.55,\ \ 1.55,\ \ 1,\ \ 20]$, $\mathbf{d}^U = [1.9,\ 1.9,\ 20,\ 80]$, $C_{p1} = \$\,3.5 \times 10^3$, $C_{p2} = \$\,1.2 \times 10^4$, $C_{\min} = \$\,1 \times 10^3$, $\gamma_1 = 0.99$, $\gamma_2 = 0.9$, $p_1 = 0.8$, and $p_2 = 0.2$.

The initial design points are $\mathbf{d}^0 = [\mu^0_{S_1},\ \mu^0_{S_2},\ N^0,\ T^0_W] = [1.65\ \text{in}^2,\ 1.65\ \text{in}^2,\ 7\ \text{times},\ 40\ \text{months}]$.

Table 5. Optimal design variables.

Optimized results				
μ_{S_1}	μ_{S_2}	T_W	N	C_{unit}
1.5715 in^2	1.8373 in^2	20 months	10.81	\$6.8628 $\times 10^3$

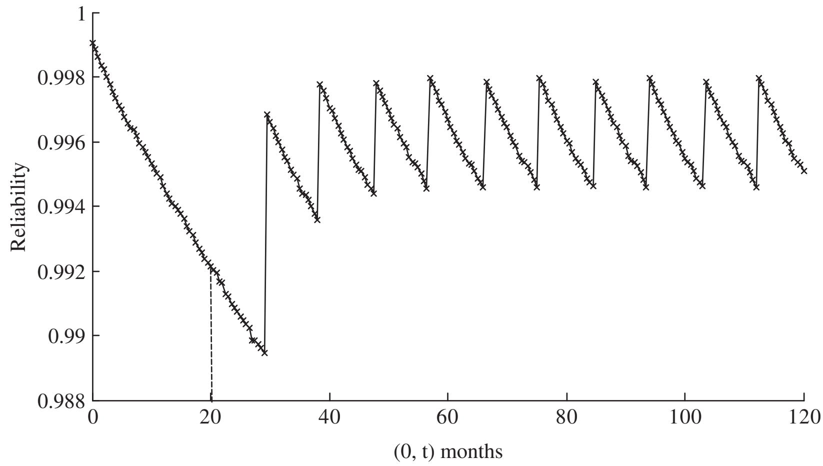

Figure 9. Time-dependent system reliability under the optimal design.

The optimization results of Example 2 are given in Table 5. The results imply that the system is planned to have a warranty period of 20 months and will be maintained 10 times during the post-warranty period. With the optimal design results, the expected unit lifecycle cost is \$6.8628 $\times 10^3$.

Figure 9 plots the time-dependent system reliability during the optimal warranty period and preventive maintenance period.

8. Conclusion

A lifetime cost optimization model is proposed to account for reliability, warranty and preventive maintenance. Physics-based time-dependent system reliability is used to predict the warranty and maintenance costs and to minimize the product lifecycle cost. The proposed method produces optimal design variables, including the optimal warranty period and optimal preventive maintenance policy.

The model extends a newly developed time-dependent component reliability method to system reliability analysis. It integrates the reliability model with the lifecycle cost model and the design optimization model. These models are coupled together by shared input variables. The two numerical examples demonstrate that the proposed model is applicable not only for the component-level, but also for the system-level design optimization.

Since the proposed model embeds the sampling-based time-dependent reliability analysis into the optimization framework, the program will call the limit-state functions repeatedly during optimization. The computational effort for solving the optimization model is relatively high. Future work is therefore needed to improve the computational efficiency. Other future work could include more stochastic processes in the optimization because the present method can handle only one stochastic process as the input to a limit-state function.

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Appendix 1. List of nomenclature

C_{life}	Expected total lifetime cost
C_{min}	Expected minimal maintenance cost per time
C_{p1}	Expected cost for preventive maintenance
C_{p2}	Expected cost for corrective maintenance
$C_{\text{pa},i}$	Expected maintenance costs of transition i for Path a
$C_{\text{pa},i,j}$	Expected maintenance cost for Path a of transition i given that system reaches the transition by the jth path
$C_{\text{pb},i}$	Expected maintenance costs of transition i for Path b
$C_{\text{pb},i,j}$	Expected maintenance cost for Path b of transition i given that system reaches the transition by the jth path
$C_{\text{p},i}$	Expected maintenance cost for transition i
$C_{\text{pu},i,j}$	Expected maintenance cost for transition i given that system reaches the transition by the jth path
C_{unit}	Expected unit lifetime cost
C_{WI}	Expected costs for type I failures over the warranty period
C_{WII}	Expected costs for type II failures over the warranty period
$P_{\text{a},i}$	Probability that the system has transition i by Path a
$P_{\text{b},i}$	Probability that the system has transition i by Path b
$P_{\text{a},i,j}$	Probability that the system reaches Path a of its ith maintenance by the jth path
$P_{\text{b},i,j}$	Probability that the system reaches Path b of its ith maintenance by the jth path
$P_{i,j}$	Probability that the system reaches its ith maintenance by the jth path
$R_{i,0}$	Residual reliability of component i at the initial time instant t_0
$R_{I,\,0}$	Initial system reliability after the warranty period
$R_{\text{I},\,i}$	Immediate system reliability after the ith maintenance
$R_{\text{I},\,i-1,j}$	Immediate system reliability after the $(i-1)$th maintenance by the jth path
$R_{s,i-1}$	System reliability right before the ith maintenance
$R_{\text{s},i-1,j}$	System reliability right before the ith maintenance by the jth path
T_{d}	Design life of product
T_{life}	Expected product service life
$T_{p,i}$	Expected waiting time between the $(i-1)$th and the ith transitions
$T_{\text{pu},i,j}$	Expected waiting time for transition i given that system reaches the transition by the jth path
T_{W}	Warranty period of product
W_i	Waiting time between two successive type II failures
W_i^*	Waiting time between two successive maintenances
γ_1	Coefficient used to quantify the capacity of corrective maintenances
γ_2	Coefficient used to quantify the capacity of preventive maintenances