Final paper to Engineering Optimization

Reliability-Based Design Optimization Under Stationary Stochastic Process Loads

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Abstract

Time-dependent reliability-based design ensures the satisfaction of reliability requirements for a given period of time, but with a high computational cost. This work improves the computational efficiency by extending the sequential optimization and reliability analysis (SORA) method to time-dependent problems with both stationary stochastic process loads and random variables. The challenge of the extension is the identification of the most probable points (MPP) associated with time-dependent reliability targets. Since the direct relationship between the MPP and reliability target does not exist, this work defines the concept of equivalent MPP, which is identified by the extreme value analysis and the inverse Saddlepoint approximation. With the equivalent MPP, the time-dependent reliability-based design optimization is decomposed into two decoupled loops: deterministic design optimization and reliability analysis, and both are performed sequentially. Two numerical examples are used to show the efficiency of the proposed method.

Keywords: Time-dependent reliability; optimization; stochastic process;

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1. Introduction

Time-dependent reliability $R(t)$ provides the probability that a system or component works properly after it has been put into operation for a period of time $[0, t]$. Let $\mathbf{X} = [X_1, X_2, \cdots, X_n]$ be a vector of random variables, $\mathbf{x} = [x_1, x_2, \cdots, x_n]$ be a realization of $\mathbf{X}$, and $\mathbf{Y}(t) = [Y_1(t), Y_2(t), \cdots, Y_m(t)]$ be a vector of stochastic processes, and then $R(t)$ is defined by

$$R(t) = \Pr\{G = g(\mathbf{X}, \mathbf{Y}(\tau)) < 0, \forall \tau \in [0, t]\} \tag{1}$$

where $g(\mathbf{X}, \mathbf{Y}(\tau))$ is a limit-state function, and G is the response. $G < 0$ indicates a working state.

The time-dependent probability of failure is computed by $p_f(t) = 1 - R(t)$, or

$$p_f(t) = \Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq 0, \exists \tau \in [0, t]\} \tag{2}$$

For special problems with only random variables $\mathbf{X}$, the reliability becomes time independent or constant. A typical time-independent RBDO model is given by

$$\begin{cases} \min\limits_{(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}_R})} f(\mathbf{d}) \\ \text{s.t.} \Pr\{g_{Pi}(\mathbf{d}, \mathbf{X}) \geq 0\} \leq [p_{fi}], \ i = 1, 2, \cdots, n_p \\ \quad g_{Dj}(\mathbf{d}) \leq 0, \ \ j = 1, 2, \cdots, n_d \end{cases} \tag{3}$$

In the above model, $f(\mathbf{d})$ is the objective function, and $\mathbf{d}$ is a vector of deterministic design variables. $\mathbf{X} = [\mathbf{X}_R, \mathbf{X}_P]$ is a vector of random variables with $\mathbf{X}_R$ being random design variables and $\mathbf{X}_P$ being random parameters. The mean values of $\mathbf{X}_R$, or $\boldsymbol{\mu}_{\mathbf{X}_R}$, are usually treated as design variables. $g_P(\cdot)$ is a constraint function for which reliability is concerned, and $[p_f]$ is the permitted probability of failure. $g_{Dj}(\cdot)$ is a deterministic constraint function.

Solving the above RBDO model is time consuming because the reliability analysis for $\Pr\{g_{Pi}(\mathbf{d}, \mathbf{X}) > 0\}$ is embedded within the optimization. Many methods have been proposed to improve the computational efficiency. The commonly used methods are the sequential single-loop methods, including the efficient reliability and sensitivity analysis method (Wu and Wang 1996) and the Sequential Optimization and Reliability Analysis (SORA) method (Du and Chen 2004). These methods decouple the RBDO process into a deterministic optimization process and a reliability analysis process. The decoupled processes make RBDO more efficient.

When time-dependent uncertainties are involved (Wang and Wang 2012), the RBDO model for a period of time $[0, t]$ becomes

$$\begin{cases} \min\limits_{(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}_R})} \ f(\mathbf{d}) \\ \text{s.t.} \ \Pr\{g_{Pi}(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \geq 0, \exists \tau \in [0, t]\} \leq [p_{fi}], \ i = 1, 2, \cdots, n_p \\ \quad g_{Dj}(\mathbf{d}) \leq 0, \quad j = 1, 2, \cdots, n_d \end{cases} \tag{4}$$

Time-dependent RBDO is much more difficult than its time-independent counterpart due to two reasons. First, it is difficult to obtain accurate reliability analysis results. Developing accurate time-dependent reliability methods is still an ongoing research topic (Mourelatos et al. 2015, Wang et al. 2015, Jiang et al. 2014). A brief review about time-dependent reliability methods is available in (Hu et al. 2012). Second, RBDO is much more time consuming because of the higher computational cost of time-dependent reliability analysis (Li, Mourelatos, and Singh 2012, Singh, Mourelatos, and Li 2010, Wang and Wang 2012).

Methodologies for time-dependent RBDO have been proposed and used in many applications. For instance, Kuschel and Rackwitz (Kuschel and Rackwitz 2000) developed a structure design optimization model by using the outcrossing rate method for time-dependent reliability analysis. Mourelatos et al. (Li, Mourelatos, and Singh 2012) introduced the time-dependent reliability analysis into the lifecycle cost optimization. Wang and Wang (Wang and Wang 2012) proposed a sequential design optimization method based on a nested extreme response method. A RBDO model was also developed in (Rathod et al. 2012) for problems with reliability degradation over time.

The accuracy and efficiency of the existing time-dependent RBDO methodologies can be further improved. For example, most of the current methods imbed the reliability constraints in the optimization framework (Li, Mourelatos, and Singh 2012, Rathod et al. 2012), and this may require a large number of function evaluations. SORA is a feasible way to improve the efficiency by decoupling the reliability analysis from optimization by employing the First-Order Reliability Method (FORM). The direct application of SORA to time-dependent problems, however, may not be accurate. In this work, we extend SORA to time-dependent RBDO based on the concept of equivalent Most Probable Point (MPP).

The main contributions of this work include the following: (1) the extension of SORA so that time-dependent RBDO with stationary stochastic processes can be solved efficiently and accurately, (2) a new concept of equivalent MPP, which allows for decoupling deterministic optimization from time-dependent reliability analysis, and (3) an efficient approach to searching for the equivalent MPP.

This paper is organized as follows. The original SORA is reviewed in Section 2. The new time-dependent SORA is discussed in Section 3, followed by the numerical procedure in Section 4. Two numerical examples are given in Section 5, and conclusions are made in Section 6.

2. Review of SORA

The original SORA is for the time-independent RBDO defined in Eq. (3) (Du and Chen 2004). SORA performs RBDO with sequential cycles of deterministic optimization and reliability analysis. After an optimal design point is found from the deterministic optimization loop, FORM is performed at this point in the reliability analysis loop. The result of the reliability analysis is then used to reformulate a new deterministic optimization model for the next cycle so that the reliability will be improved. The process continues cycle by cycle till convergence (Du and Chen 2004). The deterministic optimization in the *k*-th cycle is formulated as

$$\begin{cases} \min\limits_{[\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}_R}]} f(\mathbf{d}) \\ \text{s.t. } g_{Pi}(\mathbf{d}, T(\mathbf{u}_{\text{MPP},i}^{(k-1)})) \le 0, \; i = 1, 2, \cdots, n_p \\ \quad g_{Dj}(\mathbf{d}) \le 0, \; j = 1, 2, \cdots, n_d \end{cases} \tag{5}$$

where $\mathbf{u}_{\text{MPP},i}^{(k-1)}$ is the MPP for the *i*-th probabilistic constraint from the reliability analysis in the (*k*-1)-th cycle. $T(\cdot)$ is the transformation operator given by $\mathbf{x} = T(\mathbf{u})$, where $\mathbf{u}$ is a vector of the realization of standard normal variables $\mathbf{U}$. The result of the optimization is the optimal point $[\mathbf{d}^{(k)}, \boldsymbol{\mu}_{\mathbf{X}_R}^{(k)}]$.

Then reliability analysis or the inverse MPP search is performed at $[\mathbf{d}^{(k)}, \boldsymbol{\mu}_{\mathbf{X}_R}^{(k)}]$ for each of the probabilistic constraint functions. The MPP $\mathbf{u}_{\text{MPP},i}^{(k)}$ is obtained from the following model:

$$\begin{cases} \max\limits_{\mathbf{u}_{\mathbf{X},i}} g_{Pi}(\mathbf{d}^{(k)}, T(\mathbf{u}_{\mathbf{X},i})) \\ \text{s.t. } \left\| \mathbf{u}_{\mathbf{X},i} \right\| = \beta_i \end{cases} \tag{6}$$

in which $\left\| \cdot \right\|$ stands for the norm of a vector, and β is called a reliability index and is given by

$$\beta_i = -\Phi^{-1}([p_{fi}]) \tag{7}$$

in which $\Phi^{-1}(\cdot)$ is the inverse cumulative density function (CDF) of a standard normal variable.

The MPP $\mathbf{u}_{\text{MPP},i}^{(k)}$ corresponds directly to the permitted probability of failure $[p_{fi}]$ as shown in Eq. (7). If the constraint function at $\mathbf{u}_{\text{MPP},i}^{(k)}$ is less than 0, p_{fi} will be less than $[p_{fi}]$. Therefore,

$g_{Pi}(\mathbf{d}, T(\mathbf{u}_{\text{MPP}}^{(k)})) \le 0$ in the deterministic optimization leads to the satisfaction of the reliability requirement. After the *k*-th cycle, if no convergence is reached, the (*k*+1)-th cycle is performed.

Studies (Cho and Lee 2010, Rouhi and Rais-Rohani 2013) show that SORA performs well for RBDO problems where FORM is applied. It might also be applicable for time-dependent RBDO problems. However, the application is not straightforward since there is no direct correspondence of the MPP to the permitted time-dependent probability of failure. Major modifications of SORA are needed for time-dependent RBDO. In this work, we focus on modifying SORA for time-dependent RBDO involving only stationary stochastic processes and random variables.

3. Time-Dependent SORA (t-SORA)

In this section, we first introduce the main idea of time-dependent SORA (t-SORA) and then its details.

3.1 Overview of t-SORA

In the limit-state function $g_P(\mathbf{X}, \mathbf{Y}(\tau))$, the components of $\mathbf{Y}(\tau)$ are independent stationary stochastic processes. Since the distribution of $\mathbf{Y}(\tau)$ on $[0, t]$ does not change, so does the MPP of $g_P(\mathbf{X}, \mathbf{Y}(\tau))$.

Fig. 1 shows the flowchart of t-SORA. As t-SORA inherits the features of the original SORA, the steps of the two methods are similar. The entire optimization is performed cycle by cycle till convergence. Each cycle consists of decoupled deterministic optimization and time-dependent reliability analysis. However, the major difference or challenge is that the MPP corresponding to the permitted probability of failure $[p_f]$ cannot be directly used in the deterministic optimization. The reason is explained as follows.

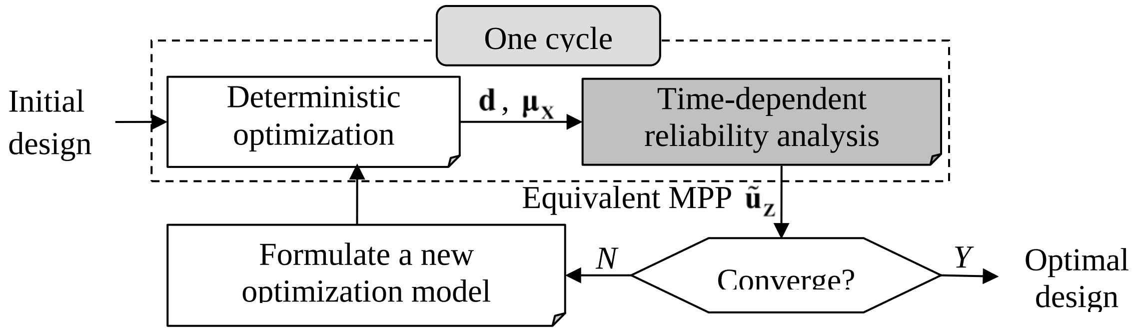

Fig. 1. Flowchart of t-SORA

With $\mathbf{Y}(\tau)$, all the random variables at τ become $\mathbf{Z}=[\mathbf{X},\mathbf{Y}(\tau)]$. The MPP $\mathbf{u}_{\mathbf{z}}^{*}$ is found by

$$\begin{cases}\max\limits_{\mathbf{u}_{\mathbf{Z}}} g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{Z}})) \\ \text{s.t.} \left\|\mathbf{u}_{\mathbf{Z}}\right\| = -\Phi^{-1}([p_{fi}])\end{cases} \tag{8}$$

where $\mathbf{u}_{\mathbf{Z}}=[\mathbf{u}_{\mathbf{X}},\mathbf{u}_{\mathbf{Y}}]$ is a vector of the realizations of standard Gaussian random variables $\mathbf{U}_{\mathbf{Z}}=[\mathbf{U}_{\mathbf{X}},\mathbf{U}_{\mathbf{Y}}]$ with $\mathbf{U}_{\mathbf{X}}$ and $\mathbf{U}_{\mathbf{Y}}$ being the transformed standard Gaussian random variables from $\mathbf{X}$ and $\mathbf{Y}(\tau)$, respectively.

Then using the original SORA, we have

$$\Pr\{g_P(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{z}}^{*}))\} = [p_{f_i}] \tag{9}$$

However, the probability $\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{z}}^{*})), \exists \tau \in [0,t]\}$ over $[0,t]$ is always greater than or equal to the time instantaneous probability $\Pr\{g_P(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{z}}^{*}))\}$ (Li, Mourelatos, and Singh 2012, Singh and Mourelatos 2010a), and therefore

$$\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{z}}^{*})), \exists \tau \in [0,t]\} \geq [p_{f_i}] \tag{10}$$

As a result, the constraint $g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{z}}^{*})) \leq 0$ in the deterministic optimization can satisfy $\Pr\{g_P(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \geq 0\} \leq [p_{f_i}]$ at only time instant τ, and it may not satisfy the time-dependent reliability requirement $\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq 0, \exists \tau \in [0,t]\} \leq [p_{f_i}]$ over the entire period of time $[0,t]$.

To address the above challenge, we propose a concept of equivalent MPP and denote it by $\tilde{\mathbf{u}}_{\mathbf{Z}}$. It is the MPP at which the limit-state function satisfies

$$\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(\mathbf{d}, T(\tilde{\mathbf{u}}_{\mathbf{Z}})), \exists \tau \in [0, t]\} = [p_{f_i}] \tag{11}$$

$\tilde{\mathbf{u}}_{\mathbf{Z}}$ can be obtained by adding the above condition to the inverse MPP search model. The new model is given by

$$\begin{cases} \max\limits_{[\mathbf{u}_{\mathbf{Z}}, \beta]} g_P(T(\mathbf{u}_{\mathbf{Z}})) \\ \text{s.t. } \|\mathbf{u}_{\mathbf{Z}}\| = \beta \\ \quad \Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(T(\tilde{\mathbf{u}}_{\mathbf{Z}})), \exists \tau \in [0, t]\} = [p_{f_i}] \end{cases} \tag{12}$$

The reliability index β is also treated as a design variable in the MPP search since it cannot be predetermined. Then the task of time-dependent reliability analysis is to search for the equivalent MPPs with Eq. (12). Doing so, however, is too computationally expensive. In this work, we develop an efficient algorithm for Eq. (12). Next, we discuss how to use the equivalent MPP in the deterministic optimization and then how to search for the equivalent MPP.

3.2 Deterministic optimization

Using the equivalent MPP, we formulate the deterministic design optimization for the *k*-th cycle as

$$\begin{cases} \min\limits_{(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}_R})} f(\mathbf{d}) \\ \text{s.t. } g_{Pi}(\mathbf{d}, T(\tilde{\mathbf{u}}_{\mathbf{Z},i}^{(k-1)}) \geq 0, i = 1, 2, \cdots, n_p \\ \quad g_{Dj}(\mathbf{d}) \leq 0, \quad j = 1, 2, \cdots, n_d \end{cases} \tag{13}$$

in which $\tilde{\mathbf{u}}_{\mathbf{Z},i}^{(k-1)}$ is the equivalent MPP for the *i*-th reliability constraint. The optimization model is similar to that of the original SORA. The only difference is that the MPPs are replaced by the equivalent MPPs. As discussed above, the use of the equivalent MPPs in constraints

$g_{Pi}(\mathbf{d}, T(\tilde{\mathbf{u}}_{\mathbf{Z},i}^{(k-1)}) \le 0, i = 1, 2, \cdots, n_p$ ensures the satisfaction of time-dependent reliability requirements.

3.3 Time-dependent reliability analysis

The task herein is to search for the equivalent MPPs for $g_P(\mathbf{X}, \mathbf{Y}(\tau)) = g_P(\mathbf{Z})$, where $\mathbf{Z} = [\mathbf{X}, \mathbf{Y}(\tau)]$, given design variables $[\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}_R}]$. As indicated in Eq. (12), there are two research issues. The first is how to calculate the time-dependent probability $\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \ge g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{Z}})), \exists \tau \in [0, t]\}$, and the second is how to solve Eq. (12) efficiently.

3.3.1 Calculation of $\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \ge g_P(\mathbf{d}, T(\mathbf{u}_{\mathbf{Z}})), \exists \tau \in [0, t]\}$

The task is to calculate the time-dependent probability $\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \ge c, \exists \tau \in [0, t]\}$, where $c = g_P(\mathbf{d}, T(\tilde{\mathbf{u}}_{\mathbf{Z}}))$ on the condition that c is known.

Time-dependent reliability methods have been extensively studied (Hu and Du 2012, Li and Mourelatos 2009, Singh, Mourelatos, and Nikolaidis 2011a, Singh and Mourelatos 2010b, Singh, Mourelatos, and Nikolaidis 2011b). Amongst them, the most commonly used one is the upcrossing rate method with the Rice's formula (Rice 1944). This method is accurate when the probability of failure is low, but may be inaccurate when the probability of failure is high. Many improvements have been made for the Rice's formula, such as considering the correlation between upcrossings (Madsen and Krenk 1984), making empirical corrections to the Rice's formula (Vanmarcke 1975), and employing important sampling (Li, Mourelatos, and Singh 2012). A brief review of the upcrossing rate method is given in Appendix A.

In this work, we use the first order sampling approach (FOSA), which completely removes the Poisson assumption used in the Rice's formula. The accuracy and efficiency are significantly improved. Details of FOSA is available in Ref. (Hu and Du 2015).

3.3.2 Equivalent MPP search

Existing MPP search algorithms cannot be used for Eq. (12) due to the constraint $\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(\mathbf{d}, T(\mathbf{u_Z})), \exists \tau \in [0,t]\} = [p_f]$. Directly solving the model in Eq. (12) will be time consuming because it involves a double-loop procedure. Using the same strategy as SORA, we propose to perform the MPP search with a sequential procedure. The idea is to separate the MPP search from the probability calculation, one is for searching for $\tilde{\mathbf{u}}_\mathbf{Z}$, and the other is for searching for β.

At first, the inverse MPP search is performed given β, with the following model:

$$\begin{cases} \max\limits_{\mathbf{u_Z}} g_P(T(\mathbf{u_Z})) \\ \text{s.t.} \left\| \mathbf{u_Z} \right\| = \beta \end{cases} \tag{14}$$

It is the regular MPP search and produces the MPP $\mathbf{u_Z}$ given β. Then the next analysis is performed to update β given $\mathbf{u_Z}$. The purpose is to find a new β so that $\Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) \geq g_P(\mathbf{d}, T(\mathbf{u_Z})), \exists \tau \in [0,t]\} = [p_f]$. How to update β will be discussed in Sec. 3.3.3. The two analyses are performed repeatedly till convergence as shown in Fig. 2. After convergence, the MPP search produces the equivalent MPP $\tilde{\mathbf{u}}_\mathbf{Z}$.

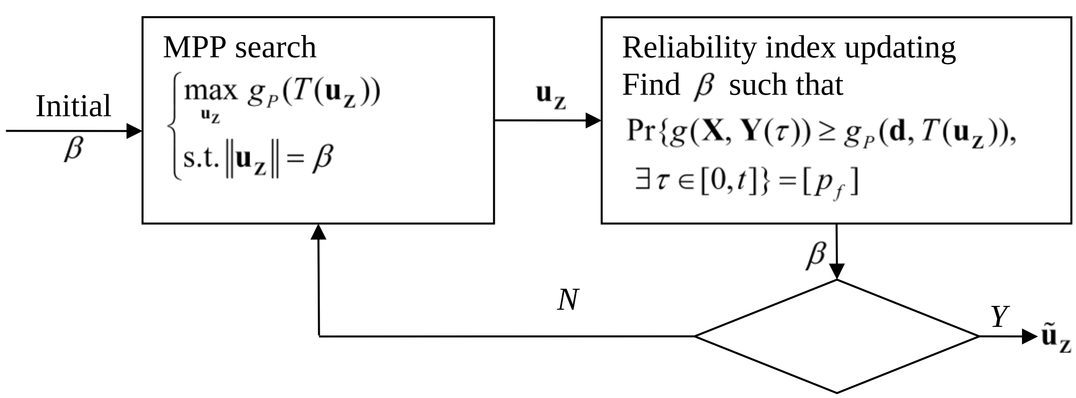

Fig. 2. Time-dependent MPP search

3.3.3 Reliability index updating

We now discuss how to update β so that $p_f(t) = \Pr\{g_P(T(\mathbf{U_Z})) \geq 0, \exists \tau \in [0,t]\} = [p_f]$. Since FORM is used, we approximate $g_P(T(\mathbf{U_Z}))$ at $\mathbf{u_Z}$. As shown in Appendix A, after approximation, the time-dependent probability is

$$p_f(t) = \Pr\{g_P(T(\mathbf{U_Z})) \geq 0\ \exists \tau \in [0,t]\} \approx \Pr\{L(\tau) = \boldsymbol{\alpha}\mathbf{U_Z}^T \geq \|\mathbf{u_Z}\|, \exists \tau \in [0,t]\} \tag{15}$$

where $L(\tau) = \boldsymbol{\alpha}\mathbf{U_Z}^T$ is a linear combination of $\mathbf{U_Z}$, and $\boldsymbol{\alpha}$ is a constant vector evaluated at $\mathbf{u_Z}$ and is given in Eq. (A2). If $p_f(t) > [\mathrm{p}_f]$, we should obtain a new reliability index β so that $p_f(t) = [p_f]$.

Let the global maximum of $L(\tau)$ over $[0,t]$ be W; namely

$$W = \max_{\tau}\{L(\tau), \tau \in [0, t]\} \tag{16}$$

$p_f(t)$ can then be calculated by

$$p_f(t) = \Pr\{W \geq \|\mathbf{u_Z}\|\} \tag{17}$$

If $p_f(t) > [\mathrm{p}_f]$, we should reduce the old reliability index $\|\mathbf{u_Z}\|$ and obtain an updated reliability index β such that

$$\Pr\{L(\tau) \geq \beta, \exists \tau \in [0,t]\} = [\mathrm{p}_f] \tag{18}$$

or

$$\Pr\{\mathrm{W} \geq \beta\} = [p_f] \tag{19}$$

It is obvious that $\beta < \|\mathbf{u_Z}\|$. The problem now becomes to find the percentile value of W given a probability level $[p_f]$. It is a difficult task because there may not be a close-form solution to the distribution of the extreme value W. Wang (Wang and Wang 2012) proposed a kriging model method to approximate the extreme value distribution, but the method is limited to limit-state functions in the form of $g(\mathbf{X}, t)$ without any input stochastic processes. Herein we use a sampling method.

Recall that $L(\tau)=\boldsymbol{\alpha}\mathbf{U}_{\mathbf{Z}}^{T}$ is a stationary Gaussian process with known coefficients $\boldsymbol{\alpha}$. We can then use simulations to obtain its sample trajectories, and for each trajectory, we find the maximum value. Then the samples of W will be available for the estimation of the CDF of W. The CDF will then produce β as indicated in Eq. (19). The samples can be efficiently generated using the Orthogonal Series Expansion (OSE) (Zhang and Ellingwood 1994), which is given in Appendix B.

Once the samples of W are available, the percentile value of W in Eq. (19) is approximated. Since $[p_f]$ is small, β is in the far right tail of the distribution of W. To obtain an accurate result, we use the saddlepoint approximation (SPA) method (HU and Du 2013). The details are provided in Appendix C. Since the sampling approach is based on $L(\tau)=\boldsymbol{\alpha}\mathbf{U}_{\mathbf{Z}}^{T}$, the original limit-state function $g_P(\cdot)$ will not be called.

3.3.4 Numerical procedure of the time-dependent reliability analysis

We now summarize the strategy of the time-dependent reliability analysis and its procedure. For a general limit-state function $g_P(\mathbf{X}, \mathbf{Y}(\tau))=g_P(\mathbf{Z})$, when it is approximated at an MPP, a number of iterations are needed to solve the following model:

$$\begin{cases} \max\limits_{[\mathbf{u}_{\mathbf{Z}},\beta]} g_P(T(\mathbf{u}_{\mathbf{Z}})) \\ \text{s.t.} \left\|\mathbf{u}_{\mathbf{Z}}\right\|=\beta \\ \quad \Pr\{\mathrm{W}\geq\beta\}=[p_f] \end{cases} \tag{20}$$

It is derived from the original model in Eq. (12) when FORM is employed. The model is solved with the procedure shown in Fig. 2 where the MPP search and reliability index updating are performed separately and sequentially. The main steps are as follows:

Step 1: Initialization: set the initial reliability index β. Since the time-dependent probability of failure is usually larger than the time-independent ones, based on our experience, we recommend using the following initial value:

$$\beta = -1.2\Phi^{-1}([p_f]) \tag{21}$$

Step 2: MPP search: Search for the MPP using Eq. (14). The results are the MPP $\mathbf{u}_\mathbf{Z}$ and vector $\boldsymbol{\alpha}$ (given in Eq. (A2)).

Step 3: Update the reliability index: (1) Construct $L(\tau)$ by $L(\tau) = \boldsymbol{\alpha}\mathbf{U}_\mathbf{Z}^T$. (2) Generate samples for $L(\tau)$ over $[0, t]$. (3) Obtain the samples of the extreme value of W. (4) Use SPA to compute the reliability index β.

Step 4: Check convergence: If the difference between the current β value and previous β is larger than a predefined tolerance, repeat Steps 2 through 4; otherwise, set the equivalent MPP $\tilde{\mathbf{u}}_\mathbf{Z} = \mathbf{u}_\mathbf{Z}$ and stop. The convergence tolerance can be set to 0.01 or 0.001 or other small numbers.

A more detailed flowchart is given in Fig. 3. The above procedure is for a general limit-state function. It should be executed for all the limit-state functions in the overall RBDO model.

4. Summary of the Numerical Procedure

We now summarize the procedure of the entire RBDO and show it in Fig. 4.

Step 1: Initialize parameters. (1) Define the initial design variables. (2) Set $k = 1$.

Step 2: Perform deterministic optimization. If $k = 1$, solve deterministic optimization at mean values of random variables and main functions of stochastic processes. If $k > 1$, formulate the deterministic optimization model using the equivalent MPPs $\tilde{\mathbf{u}}_{\mathbf{Z},i}^{(k-1)}$, where $i = 1, 2, \ldots, n_p$, obtained from the ($k-1$)-th cycle; then solve the optimization model given in Eq. (13). The optimal solution is $[\mathbf{d}^{(k)}, \mu_X^{(k)}]$.

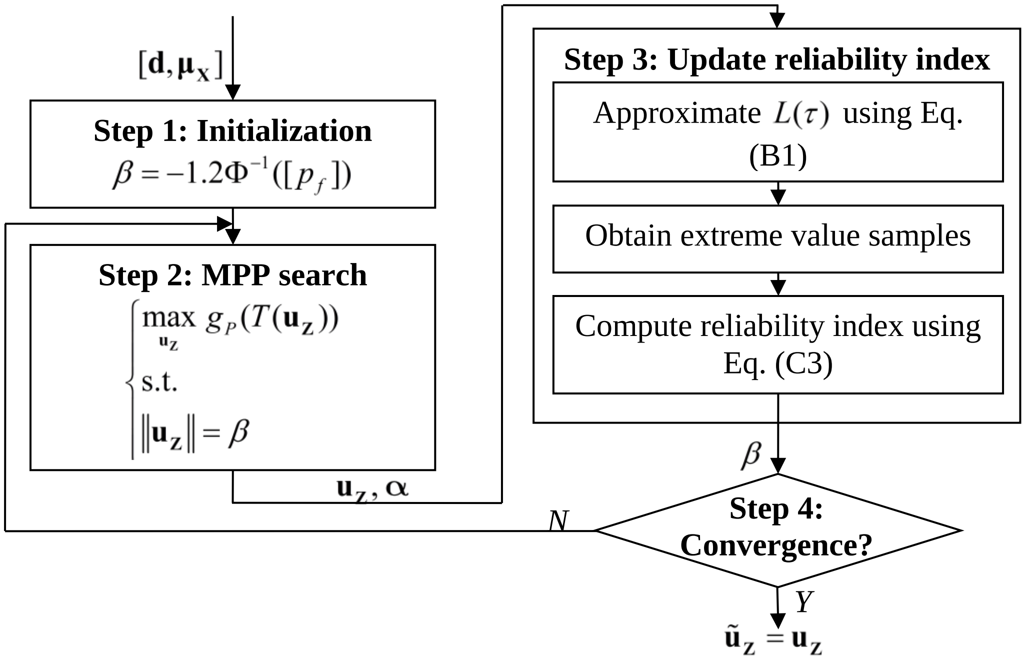

Fig. 3. Detailed flowchart for time-dependent reliability analysis

Step 3: Perform time-dependent reliability analysis at $[\mathbf{d}^{(k)}, \mu_X^{(k)}]$ following the procedure in Fig. 3. The solution is the equivalent MPPs $\tilde{\mathbf{u}}_{\mathbf{Z},i}^{(k)}$, where $i = 1, 2, \ldots, n_p$.

Step 4: Check convergence. If the limit-state functions satisfy

$$g_{Pi}(\mathbf{d}^{(k)}, T(\tilde{\mathbf{u}}_{\mathbf{Z},i}^{(k)})) \le \varepsilon \tag{22}$$

where ε is a small positive number, then the optimal solution is found and stop. Otherwise, update the cycle counter by $k = k+1$, and repeat Steps 2 through 4.

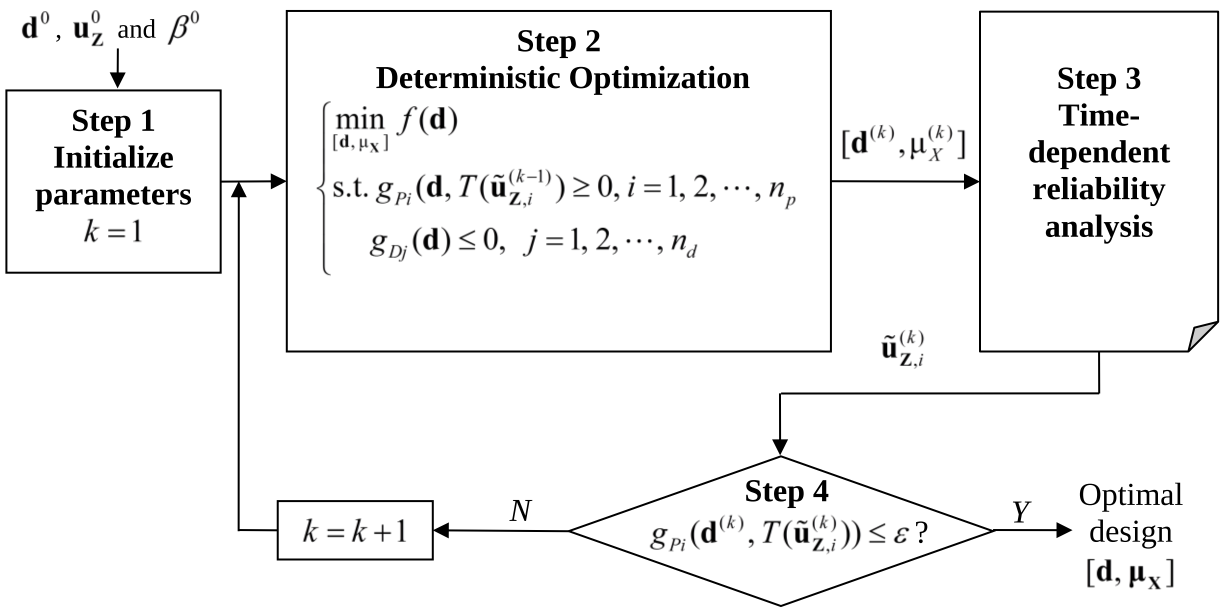

Fig. 4. Numerical procedure of t-SORA

Similar to the original SORA, the t-SORA usually converges within a few cycles, and the typical number of cycles is between three and five. In addition to the decoupling between optimization and reliability analysis, the proposed approach to the equivalent MPP search converges quickly, and this also makes t-SORA fast.

5. Numerical Examples

5.1 A two-bar frame under a stochastic force

A two-bar frame is subjected to a stochastic force $F(t)$ as shown in Fig. 5. The distances O_1O_3 and O_1O_2 are random parameters and are denoted by l_1 and l_2, respectively. Failures occur when the maximum stresses of the two bars are larger than their material yield strengths S_1 and S_2. The diameters D_1 and D_2 of the two bars are random design variables.

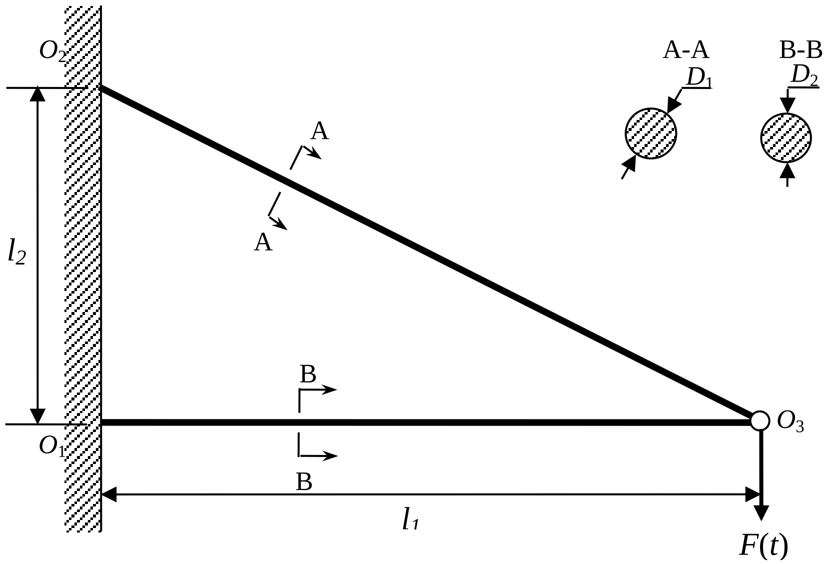

Fig. 5. A two-bar frame under stochastic force

The limit-state functions are given by

$$g_1(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) = 4F(\tau)\sqrt{l_1^2 + l_2^2} \Big/ (l_2 \pi D_1^2 S_1) - 1 \tag{23}$$

$$g_2(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) = 4F(\tau) l_1 \Big/ (l_2 \pi D_2^2 S_2) - 1 \tag{24}$$

where $\mathbf{X} = [\mathbf{X}_R, \mathbf{X}_P]$, $\mathbf{X}_R = [D_1, D_2]$, $\mathbf{X}_P = [l_1, l_2, S_1, S_2]$, $\mathbf{d} = [\mu_{D_1}, \mu_{D_2}]$, and $\mathbf{Y}(t) = [F(t)]$.

The information known is given in Table 1, where STD and GP stand for a standard deviation and a Gaussian process, respectively. The auto-correlation coefficient functions of $F(\tau)$ is

$$\rho^F(\tau_1, \tau_2) = \exp\left\{-\left((\tau_2 - \tau_1)/\zeta\right)^2\right\} \tag{25}$$

in which $\zeta = 0.1\,\text{year}$ is the correlation length.

The objective is to minimize the weight of the two bars, and the RBDO model for a service period of $[0,10]$ years is formulated as

$$\begin{cases} \min_{(\mathbf{d}, \boldsymbol{\mu}_\mathbf{X})} f(\mathbf{d}) = \pi\mu_{l_1}\mu_{D_1}^2 / 4 + \pi\sqrt{\mu_{l_1}^2 + \mu_{l_2}^2}\,\mu_{D_2}^2 / 4 \\ \text{s.t. } \Pr\{g_i(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \geq 0, \exists\tau \in [0, t]\} \leq [p_{fi}],\ i = 1, 2 \\ \quad 0.07 \text{ m} \leq \mu_{D_i} \leq 0.25 \text{ m}, \forall i = 1, 2 \end{cases} \tag{26}$$

where $[p_{f1}] = 0.01$, $[p_{f2}] = 0.001$, and $t = 10$ years.

Table 1. Random variables and stochastic process

Variable	Mean	STD	Distribution	Autocorrelation
D_1	μ_{D_1}	1×10^{-3} m	Normal	N/A
D_2	μ_{D_2}	1×10^{-3} m	Normal	N/A
S_1	1.7×10^{8} Pa	1.7×10^{7} Pa	Lognormal	N/A
S_2	1.7×10^{8} Pa	1.7×10^{7} Pa	Lognormal	N/A
l_1	0.4 m	1×10^{-3} m	Normal	N/A
l_2	0.3 m	1×10^{-3} m	Normal	N/A
$F(t)$	2.2×10^{6} N	2×10^{5} N	GP	Eq. (25)

To evaluate the accuracy and efficiency of t-SORA, we used three methods to solve the problem with the same starting point. The three methods are the t-SORA with the Orthogonal Series Expansion (OSE) method presented in Appendix B, the double-loop method using the same time-dependent reliability analysis method as t-SORA, and the double-loop method with the Rice's formula for time-dependent reliability analysis presented in Appendix A. Next we call the latter two methods the Double (OSE) and Double (Rice).

The parameters of OSE used by t-SORA and Double (OSE) are given below.

- The number of time instants that divide $[0, t]$ equally: $Q = 500$
- The number of samples generated at each time instant: $N = 10^6$
- The number of terms used in the OSE model: $M = 200$

Table 2 shows the convergence history of t-SORA. The optimal solution was obtained within three cycles. After the first cycle, the two limit-state functions were much larger than zero, and this is the indication that the reliability requirements were not met. After the third cycle, the two limit-state functions were close to zero. Then the time-dependent probabilities of failure were almost at their target values.

Table 2 Convergence history of t-SORA

k	$f\,(\mathrm{m}^3)$	(μ_{D_1}, μ_{D_2}) (m)	β_1	β_2	$g_{P1}(\mathbf{d}, T(\tilde{\mathbf{u}}_{\mathbf{Z},1}))$	$g_{P2}(\mathbf{d}, T(\tilde{\mathbf{u}}_{\mathbf{Z},2}))$
1	0.0173	(0.0831, 0.0743)	3.5662	4.2095	0.6049	0.7541
2	0.0290	(0.1051, 0.0981)	3.5715	4.2111	9.44×10^{-4}	9.24×10^{-4}
3	0.0290	(0.1051, 0.0982)	3.5721	4.2138	2.50×10^{-4}	4.88×10^{-4}

Table 3 shows the final results from the three methods. We use the number of function calls (NOFC) to measure the efficiency. t-SORA and Double (OSE) produced almost identical results. t-SORA is much more efficient than the Double (OSE) and Double (Rice) methods. The fourth and fifth columns of Table 3 present the probabilities of failure after the optimization. Since t-SORA does not compute the probabilities of failure directly, their values are not available. The probabilities of failure of the Double (OSE) and Double (Rice) methods are computed by the OSE-based sampling method (Appendix B) and Rice's formula (Appendix A), respectively. The results show that the reliability constraints were satisfied by the three optimization methods.

Table 3 Optimal results

Method	$f\,(\mathrm{m}^3)$	(μ_{D_1}, μ_{D_2}) (m)	$p_{f1}(t)$	$p_{f2}(t)$	NOFC
t-SORA	0.0290	(0.1051, 0.0982)	N/A	N/A	715
Double (OSE)	0.0290	(0.1051, 0.0982)	0.0099	0.0010	18840
Double (Rice)	0.0297	(0.1066, 0.0990)	0.0100	0.0010	11050

To verify the accuracy, we also performed Monte Carlo Simulation at the optimal design points in Table 3 from the three methods. In MCS, $[0, t]$ was discretized into 200 time instants, and 10^6 samples were generated at each time instants. Table 4 gives the percentage errors, and

Table 5 gives the 95% confidence intervals of the MCS solutions. The percentage error is computed by

$$\varepsilon = \left| p_f(t) - p_f^{\mathrm{MCS}}(t) \right| / p_f^{\mathrm{MCS}}(t) \times 100\% \tag{27}$$

For t-SORA and Double (OSE), $p_f(t)$ is calculated by the OSE-based sampling method, and for Double (Rice), it is calculated by the method based on Rice's method. $p_f^{\mathrm{MCS}}(t)$ is the probability of failure obtained from MCS.

Table 4 Accuracy comparison

	$p_{f1}(t)$	$p_{f1}^{\mathrm{MCS}}(t)$	Error (%)	$p_{f2}(t)$	$p_{f2}^{\mathrm{MCS}}(t)$	Error (%)
t-SORA	0.01	0.0094	5.3	0.001	9.4×10^{-4}	6.38
Double (OSE)	0.01	0.0094	5.3	0.001	9.4×10^{-4}	6.38
Double (Rice)	0.01	0.0046	117.39	0.001	5.4×10^{-4}	85.19

Table 5 95% confidence intervals of MCS solutions

	t-SORA	Double (OSE)	Double (Rice)
p_{f1}^{MCS}	[0.0092, 0.0096]	[0.0092, 0.0096]	[0.0044, 0.0047]
p_{f2}^{MCS}	[8.77, 9.97] $\times10^{-4}$	[8.77, 9.97] $\times10^{-4}$	[4.96, 5.88]$\times10^{-4}$

The results indicate that the accuracy of t-SORA is good. For the t-SORA and Double (OSE) methods, at the optimal design points, the actual time-dependent probabilities of failure are very close to the permitted ones. The probabilities of failure from the Double (Rice) method are much lower than the permitted ones. The reason is that the Rice's formula overestimates the probability of failure, which resulted in an over-design for this problem.

5.2 A simply supported beam under stochastic loadings

A simply supported beam shown in Fig. 6 is subjected to two stochastic loadings, which are the stochastic force $F(t)$, and the uniformly distributed loading $q(t)$. The height a and width b of the cross section are random design variables. A failure of the beam occurs when the stress exceeds the ultimate strength of the material S . The weight of the beam is expected to be minimized under the constraint that the time-dependent probability of failure of the beam over 30 years is less than 0.05. The limit-state function of the beam is given by

$$g(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) = 4\left(F(\tau)l / s + q(\tau)l^2 / 8 + \rho_{st} abl^2 / 8\right) \Big/ (ab^2 S) - 1 \tag{28}$$

where $\mathbf{X} = [\mathbf{X}_R, \mathbf{X}_P]$, $\mathbf{X}_R = [a, b]$, $\mathbf{X}_P = [S]$, $\mathbf{d} = [\mu_a, \mu_b]$, and $\mathbf{Y}(\tau) = [F(\tau), q(\tau)]$, S is the ultimate strength, ρ_{st} is the density, and l is the length of the beam.

Table 6 gives the random variables, parameters, and stochastic processes. The auto-correlation coefficient functions of $F(\tau)$ and $q(\tau)$ are

$$\rho_F(\tau_1, \tau_2) = \exp\left\{-\left((\tau_2 - \tau_1) / \zeta\right)^2\right\} \tag{29}$$

and

$$\rho_q(\tau_1, \tau_2) = \cos(\pi(\tau_2 - \tau_1)) \tag{30}$$

respectively, where $\zeta = 0.8$ year is the correlation length of $F(\tau)$.

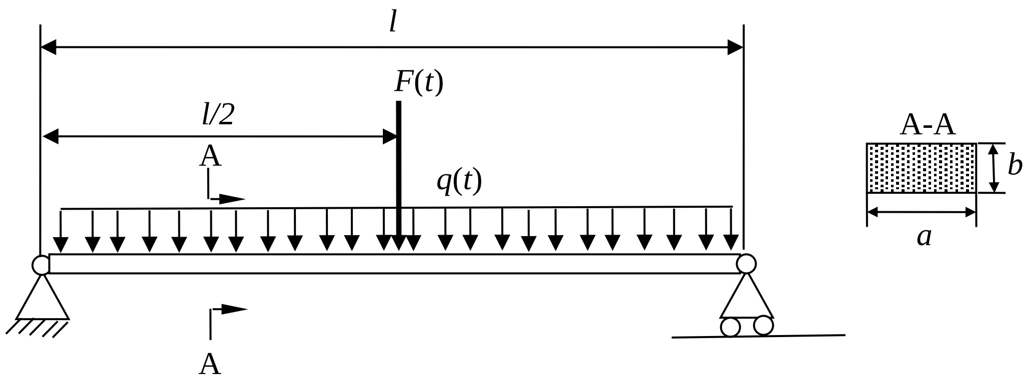

Fig. 6. A beam under stochastic loadings

The RBDO model is given by

$$\begin{cases} \min\limits_{[\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}]} f(\mathbf{d}) = \mu_a \mu_b \\ \text{s.t. } \Pr\{g(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \geq 0, \exists \tau \in [0, t]\} \leq [p_f] \\ \quad \mu_b \leq 4\mu_a ; 0.04 \text{ m} \leq \mu_a \leq 0.15 \text{ m} \\ \quad 0.15 \text{ m} \leq \mu_b \leq 0.25 \text{ m} \end{cases} \tag{31}$$

where $[p_f] = 0.05$ and $t = 30$ years.

The RBDO model was solved by the t-SORA, Double (OSE), and Double (Rice) methods with the same initial design point. Table 7 gives the convergence history of t-SORA. The results show that t-SORA converged with three cycles.

Table 6 Variables, parameters, and stochastic processes

Variable	Mean	STD	Distribution	Autocorrelation
a	μ_a	5×10^{-3} m	Lognormal	N/A
b	μ_b	5×10^{-3} m	Lognormal	N/A
S	2.4×10^{8} Pa	2.4×10^{7} Pa	Lognormal	N/A
$F(\tau)$	6000 N	600 N	GP	Eq. (29)
$q(\tau)$	900 N/m	90 N/m	GP	Eq. (30)
l	15 m	N/A	Deterministic	N/A
ρ_{st}	78.5 kN/m^3	N/A	Deterministic	N/A

Table 7 Convergence history of t-SORA

k	f (m^2)	(μ_a, μ_b) (m)	β	$g(\mathbf{d}, T(\tilde{\mathbf{u}}_{\mathbf{Z},1}))$
1	0.0065	(0.0403, 0.1613)	2.2726	0.4384
2	0.0085	(0.0460, 0.1840)	2.2887	0.0036
3	0.0085	(0.0461, 0.1842)	2.2887	1.29×10^{-5}

Table 8 presents the final results from the three methods. The results show that t-SORA is much more efficient than the other two methods.

Table 8 Optimal results

Method	f (m^2)	(μ_a, μ_b) (m)	$p_f(t)$	NOFC

t-SORA	0.0085	(0.0461, 0.1842)	N/A	156
Double (OSE)	0.0085	(0.0463, 0.1836)	0.0500	7756
Double (Rice)	0.0092	(0.0478, 0.1914)	0.0500	1612

Table 9 gives the probabilities of failure from MCS at the optimal design points from the three aforementioned methods. Table 10 presents the 95% confidence intervals of the MCS solutions. The time interval [0, 30] year was divided into 600 time instants, and 10^6 samples were generated at each time instant for MCS. The t-SORA and Double (OSE) methods are more accurate than the Rice's formula, which overestimated the probability of failure. The optimal design obtained from the Double (Rice) method is therefore conservative.

Table 9 Accuracy comparison

	$p_f(t)$	$p_f^{\mathrm{MCS}}(t)$	Error (%)
t-SORA	0.05	0.0522	4.2
Double (OSE)	0.05	0.0522	4.2
Double (Rice)	0.05	0.0093	440.96

Table 10 95% confidence intervals of MCS solutions

	t-SORA	Double (OSE)	Double (Rice)
$p_f^{\mathrm{MCS}}(t)$	[0.0517 0.0526]	[0.0518 0.0527]	[0.0091 0.0094]

Although the two examples involve explicit limit-state functions, they are actually treated as black-box functions since the derivatives of the limit-state functions are evaluated numerically.

6. Conclusion

Time-dependent Sequential Optimization and Reliability Analysis (t-SORA) method is developed for problems with both random variables and stochastic processes. To address the limitation that there is no direct connection between time-dependent reliability and the Most Probable Point (MPP), t-SORA uses the equivalent MPP, which directly corresponds to the required time-dependent reliability. This ensures that the overall optimization be solved sequentially in cycles of deterministic optimization and reliability analysis. The results show that t-SORA can effectively solve design optimization with time-dependent reliability constraints.

The proposed method is based on FORM. Its accuracy is then affected by the linearization made by FORM. However, the proposed method may not be limited to FORM. If the limit-state function in the transformed normal variable space is highly nonlinear, more accurate reliability analysis methods, such as the Second Order Reliability Method (SORM), can also be used.

t-SORA is for problems with only stationary stochastic processes. The same strategy may be extended to problems with non-stationary processes. The implementation, however, will be significantly different and will be more challenging. It needs a further investigation.

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Appendix A. Reliability analysis with the Rice's formula and FORM

For a limit-state function $g_P(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) = g_P(\mathbf{d}, \mathbf{Z}(t))$, where $\mathbf{Z}(t) = [\mathbf{X}, \mathbf{Y}(t)]$, its MPP is obtained from

$$\begin{cases} \min \|\mathbf{U}_{\mathbf{Z}}\| \\ \text{s.t. } g_P(T[\mathbf{U}_{\mathbf{Z}}(t)]) \geq 0 \end{cases} \tag{A1}$$

where $\mathbf{U}_{\mathbf{Z}}(t) = [\mathbf{U}_X, \mathbf{U}_Y(t)]$ is the standard normal variables associated with $\mathbf{X}$ and $\mathbf{Y}(t)$.

After the MPP $\mathbf{u}(t)=[\mathbf{u}_{\mathbf{X}}, \mathbf{u}_Y(t)]$ is found, the limit-state function $g_P(\mathbf{d}, \mathbf{X}, \mathbf{Y}(t)) = g_P(\mathbf{d}, \mathbf{Z}(t))$ is linearized at the MPP, the time-dependent probability of failure given in Eq. (5) is then approximated by (Hu and Du 2012, Hagen and Tvedt 1991):

$$p_f(t) = \Pr\{g_P(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \geq 0, \exists \tau \in [0, t]\} \approx \Pr\{L(\tau) = \boldsymbol{\alpha}\mathbf{U}_{\mathbf{Z}}^T(\tau) > \beta(\tau), \exists \tau \in [0, t]\} \quad \text{(A2)}$$

in which $\beta(\tau) = \|\mathbf{u}(\tau)\|$ and $\boldsymbol{\alpha} = \mathbf{u}(\tau)/\beta$.

The Rice's formula gives the upcrossing rate by (Rice 1944)

$$v^+(t) = \omega(t)\phi(\beta(t))\Psi\left(\dot{\beta}(t)/\omega(t)\right) \quad \text{(A3)}$$

where $\phi(\cdot)$ is the probability density function (PDF) of a standard normal variable, and

$$\dot{\beta}(t) = \partial\beta(t)/\partial t \quad \text{(A4)}$$

$$\Psi(x) = \phi(x) - x\Phi(-x) \quad \text{(A5)}$$

and

$$\omega^2(t) = \dot{\boldsymbol{\alpha}}\dot{\boldsymbol{\alpha}}^T + \boldsymbol{\alpha}\ddot{\mathbf{C}}_{12}(t, t)\boldsymbol{\alpha}^T \quad \text{(A6)}$$

in which

$$\dot{\boldsymbol{\alpha}} = \partial\boldsymbol{\alpha}(t)/\partial t \quad \text{(A7)}$$

and

$$\ddot{\mathbf{C}}_{12}(t_1, t_2)\Big|_{t_1=t_2=t} = \left.\begin{bmatrix} \mathbf{0} & 0 & 0 & \cdots & 0 \\ 0 & \partial^2\rho_1(t_1, t_2)/\partial t_1 \partial t_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \partial^2\rho_m(t_1, t_2)/\partial t_1 \partial t_2 \end{bmatrix}\right|_{t_1=t_2=t} \quad \text{(A8)}$$

in which $\rho_l(t_1, t_2)$, $l = 1, 2, \cdots m$, are the coefficients of the autocorrelation of stochastic process $U_{Y_l}(t)$, and m is the number of stochastic processes. Since the stochastic processes $\mathbf{Y}(t)$ are assumed to be stationary, $\dot{\boldsymbol{\alpha}} = 0$, and $\dot{\beta} = 0$.

p_f is computed by (Hu and Du 2012)

$$p_f(\mathrm{t}) = 1 - R(0)\exp\left\{-\left(\int_0^t v^+(\tau)d\tau\right)\right\} \tag{A9}$$

where $R(0) = \Phi(\beta)$ is the time instantaneous reliability at the initial time instant.

Appendix B. Orthogonal Series Expansion (OSE)

As shown in Eq. (A2), the time-dependent probability of failure $p_f(t)$ is approximated by $p_f(t) = \Pr\{G(\tau) = g_P(\mathbf{d}, \mathbf{X}, \mathbf{Y}(\tau)) \ge 0, \exists \tau \in [0, t]\} \approx \Pr\{L(\tau) = \boldsymbol{\alpha}\mathbf{U}_{\mathbf{Z}}^T(\tau) \ge \beta, \exists \tau \in [0, t]\}$, where $G(\tau)$ is a non-Gaussian stochastic process and $L(\tau)$ is a standard Gaussian stochastic process. If the maximum value of $L(\tau)$ over $[0, t]$, W , is available, according to Eq. (17), $p_f(t) = \Pr\{W \ge \beta\}$. The distribution of W can be obtained from the samples of $L(\tau)$, and the samples may be generated from the OSE method.

OSE approximates $L(\tau)$ as follows (Zhang and Ellingwood 1994):

$$L(\tau) \approx \sum_{i=0}^{M} \sqrt{\lambda_i}\xi_i \left(\sum_{j=0}^{M} P_j^i h_j(\tau)\right) \tag{B1}$$

in which λ_i is the i-th eigenvalue of covariance matrix $\boldsymbol{\Sigma}$, P_j^i is the projection of the i-th eigenvector of covariance matrix $\boldsymbol{\Sigma}$ on the j-th Legendre polynomial, and $h_j(t)$ is the j-th Legendre polynomial, ξ_i, where $i = 1, 2, \cdots, M$, are M independent standard normal variables, and $\boldsymbol{\Sigma}$ is a matrix with element Σ_{ij} given by (Zhang and Ellingwood 1994)

$$\Sigma_{ij} = \int_0^t \int_0^t \rho_{t_1 t_2} h_i(t_1) h_j(t_2) dt_1 dt_2 \tag{B2}$$

where

$$\rho_{\tau_1\tau_2} = \boldsymbol{\alpha}\mathbf{C}(\tau_1, \tau_2)\boldsymbol{\alpha}^T \tag{B3}$$

and $\mathbf{C}(\tau_1, \tau_2)$ is a diagonal matrix with the diagonal element being the covariance of $\mathbf{U_Z}(\tau_1)$ and $\mathbf{U_Z}^T(\tau_2)$.

Once the approximated response $L(\tau)$ is available, N samples can be generated at Q discretizing instants over $[0, t]$. The samples are given in matrix $\tilde{L}_{N\times Q}$ as below.

$$\tilde{L}_{N\times Q} = \begin{pmatrix} l(t_1, 1) & l(t_2, 1) & \cdots & l(t_Q, 1) \\ l(t_1, 2) & l(t_2, 2) & \cdots & l(t_Q, 2) \\ \vdots & \vdots & \ddots & \vdots \\ l(t_1, N) & l(t_2, N) & \cdots & l(t_Q, N) \end{pmatrix}_{N\times Q} \tag{B4}$$

N samples of the extreme value W can then available through the following equations:

$$w_j = \max\{l(t_1, j), l(t_2, j), \cdots, l(t_Q, j)\}, \text{ where } j = 1, 2, \cdots, N \tag{B5}$$

Appendix C. Saddlepoint Approximation (SPA)

At first, the cumulants of W are computed from the samples by (Fisher 1928)

$$\begin{cases} \kappa_1 = m_1 / N \\ \kappa_2 = (Nm_2 - m_1^2)/(N(N-1)) \\ \kappa_3 = (2m_1^3 - 3nm_1m_2 + N^2 m_3)/(N(N-1)(N-2)) \\ \kappa_4 = \dfrac{-6m_1^4 + 12nm_1^2 m_2 - 3N(N-1)m_2^2}{N(N-1)(N-2)(N-3)} + \dfrac{-4N(N+1)m_1m_3 + N^2(N+1)m_4}{N(N-1)(N-2)(N-3)} \end{cases} \tag{C1}$$

where κ_i is the *i*-th cumulant of W, m_s ($s = 1, 2, 3, 4$) are the sums of the *s-th* power of the samples W and are given by

$$m_s = \sum_{i=1}^{N} w_i^s \tag{C2}$$

in which w_i is the *i*-th sample of W given in Eq. (B5).

In this work, the first four moments are used. Higher order may also be used. Once κ_j, $j = 1, 2, 3, 4$, are available, the reliability index β is updated by

$$\beta = \kappa_1 + \kappa_2\,\eta/1! + \kappa_3\,\eta^2/2! + \kappa_4\,\eta^3/3! \tag{C3}$$

where η is the saddlepoint, which satisfies the following equations:

$$1 - [p_f] = \Phi(z) + \phi(z)\left(1/z - 1/v\right) \tag{C4}$$

$$z = \mathrm{sign}(\eta)\left\{2\left[\eta K_L'(\eta) - K_L(\eta)\right]\right\}^{1/2} \tag{C5}$$

$$v = \eta\left[K_L''(\eta)\right]^{1/2} \tag{C6}$$

$$K_L(\eta) = \sum_{i=1}^{4} \kappa_i\,\eta^i/i! \tag{C7}$$

and

$$K_L''(\eta) = \kappa_2 + \sum_{j=3}^{4} \kappa_j\,\eta^{j-2}/(j-2)! \tag{C8}$$

where $\mathrm{sign}(\eta) = +1, -1$, or 0, depending on whether η is positive, negative, or zero.

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