Uncertainty Quantification of Time-Dependent Reliability Analysis in the Presence of Parametric Uncertainty

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Limited data of stochastic load processes and system random variables result in uncertainty in the results of time-dependent reliability analysis. An uncertainty quantification (UQ) framework is developed in this paper for time-dependent reliability analysis in the presence of data uncertainty. The Bayesian approach is employed to model the epistemic uncertainty sources in random variables and stochastic processes. A straightforward formulation of UQ in time-dependent reliability analysis results in a double-loop implementation procedure, which is computationally expensive. This paper proposes an efficient method for the UQ of time-dependent reliability analysis by integrating the fast integration method and surrogate model method with time-dependent reliability analysis. A surrogate model is built first for the time-instantaneous conditional reliability index as a function of variables with imprecise parameters. For different realizations of the epistemic uncertainty, the associated time-instantaneous most probable points (MPPs) are then identified using the fast integration method based on the conditional reliability index surrogate without evaluating the original limit-state function. With the obtained time-instantaneous MPPs, uncertainty in the time-dependent reliability analysis is quantified. The effectiveness of the proposed method is demonstrated using a mathematical example and an engineering application example. [DOI: 10.1115/1.4032307]

Keywords: Bayesian approach, time-dependent reliability, stochastic loading, time series, epistemic uncertainty

1 Introduction

Time-dependent reliability analysis considers both the statistical variation at a time instant and variations over time. During the past decades, a group of time-dependent reliability analysis methods have been proposed [1–5]. For instance, Madsen and Tvedt presented a general and efficient method for time-dependent reliability and sensitivity analysis [2]; Mori and Ellingwood [6] proposed an important sampling approach for time-dependent system reliability analysis and performed service-life assessment for aging concrete structures using time-dependent reliability analysis [7]; Zheng and Ellingwood investigated the role of nondestructive evaluation in time-dependent reliability analysis [8]; Hagen and Tvedt [9,10] proposed a parallel system approach to solve time-dependent problems with binomial distributions; Andrieu-Renaud et al. developed a PHI2 method for problems with random variables and stochastic processes [11]; Sudret [12] derived analytical expressions for the outcrossing rate in time-dependent problems and applied the developed method to the cooling towers [13] and degradation of reinforced concrete structures [14]; and Li and Chen developed a reliability analysis method for dynamic response using a new probability density evolution approach [15].

Review of the above literature indicates that most of the current methods are based on the assumption that the random variables and stochastic processes are modeled with abundant data (i.e., no epistemic uncertainty, only aleatory variability). In practical engineering applications, however, the collected data of random variables and stochastic loadings are usually limited either because of limitations of experiments or shortage of historical data. For example, when designing a wind turbine system for a 20-year service life, the designer may have only historical wind speed data for the previous 30 or 50 years [16,17]. The limited data cause epistemic uncertainty in the modeling of the random variables and stochastic loading. Besides, noise and discrepancy in the sensors and measurement conditions also contribute to uncertainty in the data about random variables and stochastic loads. There are also other types of epistemic uncertainty sources in time-dependent reliability analysis, such as model uncertainties due to the use of model form assumptions and numerical approximations. In the presence of all these sources of uncertainties, a question that needs to be quantitatively answered is: how confident are we in our reliability analysis result?

Considering epistemic uncertainty while performing reliability analysis has gained much attention during the past decades. For example, Der Kiureghian and Liu [18] developed a framework for the analysis of structural reliability under incomplete probability information; Der Kiureghian also investigated the assessment of structural safety under imperfect states of knowledge [19] and parameter uncertainties [20]; Der Kiureghian and Ditlevsen [21] discussed the importance of considering epistemic uncertainty during reliability analysis; Roland and Sudret developed an imprecise reliability analysis method using PC-Kriging [22]; Li et al. calculated the probability of failure distribution using the Bayes' rule [23]; Wang et al. developed a Bayesian reliability analysis method for problems with insufficient and subjective data sets [24]; and Coolen and Newby developed an extension of the standard Bayesian approach for reliability analysis based on imprecise probabilities and intervals of measures [25]. Although many reliability analysis methods under epistemic uncertainty have been proposed, available methods mainly focus on the epistemic uncertainty of random variables and time-independent reliability analysis. This paper aims to develop a UQ framework that performs time-dependent reliability analysis and accounts for both aleatory and epistemic uncertainty during the analysis.

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Manuscript received March 26, 2015; final manuscript received December 04, 2015; published online Month XX, XXXX. Assoc. Editor: Ioannis Kougioumtzoglou.

In this work, the Bayesian approach is used to describe the epistemic uncertainty in both system random variables and stochastic load processes in time-dependent reliability analysis. A straightforward way of UQ in time-dependent reliability analysis is to implement a double-loop procedure. In the outer loop, realizations of epistemic variables are generated, and time-dependent reliability analysis is performed in the inner loop conditioned on the realizations of the epistemic variables. Since time-dependent reliability analysis is already very computationally expensive, the double-loop procedure is computationally prohibitive. A surrogate model is an obvious choice. But, building a surrogate model of the time-dependent failure probability as a function of epistemic parameters is still computationally expensive. This paper proposes an efficient two-step approach for the UQ in time-dependent reliability analysis. A surrogate model of the time-instantaneous conditional reliability index is built first as a function of variables with epistemic parameters. The conditional reliability index surrogate model is then integrated with the fast integration method to efficiently identify the time-instantaneous MPP under different realizations of epistemic parameters without evaluating the original limit-state function. Based on the time-instantaneous MPPs, the uncertainty in time-dependent reliability analysis is quantified. The developed method improves the efficiency of UQ of time-dependent reliability analysis significantly. In addition, this paper also investigates the time-dependent reliability analysis method by using the data-driven time-series models instead of the common practice of using stochastic process models with exact mean, variance, and correlation functions. The contributions of this paper are therefore summarized as (1) a new method to reduce the computational effort of UQ in time-dependent reliability analysis, and (2) development of a UQ framework for time-dependent reliability analysis.

In Sec. 2, backgrounds of time-dependent reliability, commonly used time-dependent reliability analysis methods, and a sampling approach are briefly reviewed. Section 3 proposes the developed UQ framework for time-dependent reliability analysis. Two numerical examples are given in Sec. 4. Conclusions are made in Sec. 5.

2 Background

2.1 Time-Dependent Reliability. Let $G(t) = g(\mathbf{X}, \mathbf{Y}(t), t)$ be a time-dependent response function, where $\mathbf{X} = [X_1, X_2, \ldots, X_n]$ is a vector of random variables; $\mathbf{Y}(t) = [Y_1(t), Y_2(t), \ldots, Y_m(t)]$ is a vector of stochastic processes; and $g(\cdot)$ is a response function, and t stands for time. The time-dependent probability of failure is given by Ref. [26]

$$p_f(t_0, t_e) = \Pr\{G(\tau) = g(\mathbf{X}, \mathbf{Y}(\tau), \tau) > e, \exists \tau \in [t_0, t_e]\} \quad (1)$$

in which e is a specific failure threshold; $\Pr\{\cdot\}$ stands for probability; "$\exists$" means "there exists"; and t_0 and t_e are the initial and final time instants, respectively.

2.2 Time-Dependent Reliability Analysis Methods. As reviewed in Sec. 1, many approaches have been proposed to efficiently estimate the time-dependent reliability analysis in past decades. Currently available methods can be roughly classified into three groups: upcrossing rate methods, sampling-based approaches, and surrogate model-based methods. The upcrossing rate methods based on the Poisson assumption [9], such as the PHI2 method, are commonly used due to their simplicity of implementation. However, these methods could result in significant errors for problems that have both random processes and random variables [27,28]. The error can be several orders of magnitude, depending on the mean number of upcrossings in the time interval of interest and the relative magnitude between the variances of load processes and random variables [29–31]. In order to release the Poisson assumption, corrections have been suggested in computing the upcrossing rate [30,32,33]. Efforts have also been made to remove the Poisson assumption, in sampling-based [34,35] and surrogate model-based [36,37] methods.

This paper focuses on quantifying the uncertainty in time-dependent reliability analysis due to the presence of data uncertainty. It is illustrated with currently available time-dependent reliability analysis methods. However, the proposed UQ approach is general and can be applied with any preferred time-dependent reliability analysis method. In this work, the first-order sampling approach (FOSA) [34] is used to illustrate the proposed framework of UQ in time-dependent reliability analysis. The presented framework in the following sections, however, is not limited to the FOSA method. It is applicable to the upcrossing rate method and other methods as well. Before discussing the proposed UQ framework, the FOSA method is briefly reviewed as follows.

2.2.1 Review of the First-Order Sampling Approach. Since time-dependent reliability analysis for nonstationary loading is computationally expensive, UQ in this case is even more computationally intensive. In this paper, we focus only on weakly stationary loading. For stationary problems with random variables and weakly stationary stochastic loading, the time-dependent probability of failure becomes

$$p_f(t_0, t_e) = \Pr\{G(\tau) = g(\mathbf{X}, \mathbf{Y}(\tau)) > e, \exists \tau \in [t_0, t_e]\} \quad (2)$$

Even for the stationary Gaussian loading process, the response $G(\tau)$ is a stationary non-Gaussian process if the response function $g(\mathbf{X}, \mathbf{Y}(\tau))$ is a nonlinear function. The basic principle of FOSA is to model the response stochastic process $G(\tau)$ directly at the output level. A weakly stationary stochastic process has the following properties: (1) the statistical properties (mean and standard deviation) do not change with time; and (2) the autocorrelation is only dependent on the distance between two time instants. Even though $G(t)$ has these special properties, directly modeling $G(t)$ is still difficult because the statistical properties of $G(t)$ are unknown. In order to overcome this difficulty, FOSA models an equivalent stochastic process $L_G(t)$ based on the following probability equivalency [34]:

$$\begin{aligned}\Pr\{G(\tau) = g(\mathbf{X}, \mathbf{Y}(\tau)) > e, \exists \tau \in [t_0, t_e]\} &= \Pr\{L_G(\tau) \\ &= \boldsymbol{\alpha}_{\mathbf{X}}\mathbf{U}_{\mathbf{X}}^T + \boldsymbol{\alpha}_{\mathbf{Y}}\mathbf{U}_{\mathbf{Y}}^T(\tau) > \beta, \exists \tau \in [t_0, t_e]\}\end{aligned} \quad (3)$$

where β is the reliability index; $L_G(t)$ is the equivalent stochastic process; $\boldsymbol{\alpha}_{\mathbf{X}} = \mathbf{u}_{\mathbf{X}}^*/\|\mathbf{u}^*(t_0)\|$; $\boldsymbol{\alpha}_{\mathbf{Y}} = \mathbf{u}_{\mathbf{Y}}^*(t_0)/\|\mathbf{u}^*(t_0)\|$; $\mathbf{U}_{\mathbf{X}}$ and $\mathbf{U}_{\mathbf{Y}}(t)$ are the standard normal variables and standard Gaussian stochastic processes corresponding to $\mathbf{X}$ and $\mathbf{Y}(t)$, respectively; and $\mathbf{u}^*(t_0) = [\mathbf{u}_{\mathbf{X}}^*, \mathbf{u}_{\mathbf{Y}}^*(t_0)]$ is the time-instantaneous MPP identified from the following optimization model:

$$\begin{cases} \min \beta(t_0) = \|\mathbf{u}(t_0)\| \\ \mathbf{u}(t_0) = [\mathbf{u}_{\mathrm{X}}, \mathbf{u}_{\mathrm{Y}}(t_0)] \\ G(t_0) = g(T(\mathbf{u}_{\mathrm{X}}), T(\mathbf{u}_{\mathrm{Y}}(t_0))) \le e \end{cases} \quad (4)$$

in which $\|\cdot\|$ is the determinant of a vector and $T(\cdot)$ is an operator, which transforms $\mathbf{u}_{\mathbf{X}}$ and $\mathbf{u}_{\mathbf{Y}}(t_0)$ into original random variables $\mathbf{X}$ and $\mathbf{Y}(t_0)$.

The equivalent stochastic process $L_G(t)$ has the following properties:

1. $L_G(t)$ is weakly stationary, since $G(t)$ is weakly stationary;
2. $L_G(t)$ is a weakly stationary Gaussian process with zero mean and unit standard deviation;
3. The autocorrelation function of $L_G(t)$ is given by [38]

$$\rho_L(t_1, t_2) = \boldsymbol{\alpha}_{\mathbf{X}}\boldsymbol{\alpha}_{\mathbf{X}}^T + \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\rho}(t_1, t_2)\boldsymbol{\alpha}_{\mathbf{Y}}^T \quad (5)$$

where

$$\boldsymbol{\rho}(t_1,t_2)=\begin{bmatrix}\boldsymbol{\rho}_{Y_1}(t_1,t_2) & 0 & \cdots & 0\\ 0 & \ddots & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{\rho}_{Y_m}(t_1,t_2)\end{bmatrix}_{m\times m} \quad (6)$$

in which $\rho_{Y_i}(t_1,t_2)$, $i=1,2,\ldots,m$, is the autocorrelation coefficient of $U_{Y_i}(t)$ between time instants t_1 and t_2.

The above analyses indicate that through only one MPP search, the statistical properties of the equivalent stochastic process $L_G(t)$ can be obtained. With the above statistical information, the equivalent stochastic process $L_G(t)$ can be modeled directly without evaluating the original limit-state function. Here, the expansion optimal linear estimation method (EOLE) method [39] is employed to model $L_G(t)$. In EOLE, $[t_0,t_e]$ is first discretized into h time instants, t_i, $i=1,2,\ldots,h$. EOLE then expands $L_G(t)$ into a finite series of random variables based on the eigenvalue and eigenvector analysis of the covariance matrix $\boldsymbol{\rho}_L$ given as follows:

$$\boldsymbol{\rho}_L=\begin{bmatrix}1 & \rho_L(t_1,t_2) & \cdots & \rho_L(t_1,t_h)\\ \rho_L(t_2,t_1) & \ddots & \cdots & \rho_L(t_2,t_h)\\ \vdots & \vdots & \ddots & \vdots\\ \rho_L(t_h,t_1) & \rho_L(t_h,t_2) & \cdots & 1\end{bmatrix}_{h\times h} \quad (7)$$

where $\rho_L(t_i,t_j)$, $i,j=1,2,\ldots,h$, are computed using Eq. (5).

Let η_i and $\boldsymbol{\varphi}_i^T$ be the eigenvalues and eigenvectors of the correlation matrix $\boldsymbol{\rho}_L$, $L_G(t)$ is then modeled using the EOLE method as below

$$L_G\approx\sum_{i=1}^{r}\frac{\xi_i}{\sqrt{\eta_i}}\boldsymbol{\varphi}_i^T\boldsymbol{\rho}_{Lt}(t),\quad \forall t\in[t_0,t_e] \quad (8)$$

where ξ_i, $i=1,2,\ldots,r$, is a vector of independent standard normal variables; $\boldsymbol{\rho}_{Lt}(t)=[\rho_L(t,t_1),\rho_L(t,t_2),\ldots,\rho_L(t,t_h)]^T$; and $r\le h$ is the number of terms of expansion. Note that the eigenvalues η_i are sorted in a decreasing order.

With the expression given in Eq. (8), samples of $L_G(t)$ are generated by discretizing $[t_0,t_s]$ into W time instants and generating N samples for each random variable of ξ_i. The number of N can be very large, since it will not evaluate the original limit-state function. In this paper, $N=2\times10^6$ is used. After that, the time-dependent probability of failure is estimated using Eq. (3) based on the samples of $L_G(t)$ over $[t_0,t_s]$. Note that two main approximations are made in FOSA for stationary problems: (1) linearization of the limit state using the first-order reliability method (FORM) (Eq. (3)), and (2) modeling of the equivalent stochastic process using the expansion method (Eq. (8)). The method is therefore applicable only to problems in which FORM is accurate for time-instantaneous reliability analysis. As mentioned earlier, the developed method is not limited to FOSA. It can also be applied to other time-dependent reliability analysis methods.

3 UQ in Time-Dependent Reliability Analysis

In the above reviewed reliability analysis method, all the random variables and stochastic processes are assumed to be accurately modeled. Only aleatory uncertainty (natural variability) is considered in the evaluation of the time-dependent reliability. In reality, there are other sources of epistemic uncertainty present due to limited information (i.e., data uncertainty and model uncertainty). Due to such epistemic uncertainty sources, the obtained time-dependent reliability analysis result is also uncertain. In this section, the epistemic uncertainty sources in time-dependent reliability analysis are analyzed first. After that, the effects of these uncertainties on the time-dependent probability of failure are quantified.

3.1 Uncertainty Sources.

The uncertainty sources that affect the results of time-dependent reliability analysis can be roughly classified into two categories:

- *Data uncertainty:* In practical applications, parameters of random variables are modeled based on the collected data. Due to noise and measurement limitations, uncertainties are inherent in the collected data. Sensor degradation and measurement conditions also cause uncertainty in the data.
- *Model uncertainty:* In reliability analysis, the response function needs to be evaluated at given design points. The response function can be a finite element analysis (FEA) model or other simulation models. These simulation models will inevitably have some errors due to model form assumption and numerical approximations. There are also model uncertainties in the models of random variables and stochastic processes.

The data uncertainty and model uncertainties in random variables and stochastic processes are the focus of this paper.

3.2 Uncertainty Modeling of Random Variables.

For some random variables, the collected data are too limited to precisely determine the distribution type or parameters of the random variables. In this situation, the Bayesian approach can be used to represent the epistemic uncertainty in both the random variable parameters and distribution-type. For a random variable X, the joint probability density function (PDF) of its parameters $\boldsymbol{\theta}$ under given observations $\mathbf{x}=[x_1,x_2,\ldots,x_{ob}]$ is updated with the Bayes' theorem as follows:

$$p(\boldsymbol{\theta}|\mathbf{x})=\frac{L(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int L(\mathbf{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}} \quad (9)$$

where $\boldsymbol{\theta}=[\theta_1,\theta_2,\ldots,\theta_{nr}]$ is a vector of parameters of the random variable; $L(\mathbf{x}|\boldsymbol{\theta})$ is the likelihood function of observations $\mathbf{x}$ given parameters $\boldsymbol{\theta}$; $\pi(\boldsymbol{\theta})$ is the prior distribution; and $p(\boldsymbol{\theta}|\mathbf{x})$ is the updated posterior distribution of $\boldsymbol{\theta}$.

Directly solving the above equation is difficult due to the involvement of the multidimensional integration. Instead, Markov chain Monte Carlo (MCMC) sampling is commonly employed to evaluate Eq. (9).

3.3 Uncertainty Modeling of Stochastic Process Loads

3.3.1 Time-Series Model for Stochastic Loading. The commonly used approaches for the stochastic modeling of loading time histories assume that the mean, standard deviation, and correlation functions or frequency spectrum of the loading are exactly known. With the known information, the stochastic loads are simulated using spectral representation methods, such as the Karhunen–Loeve (KL) expansion method, polynomial chaos expansion (PCE), the orthogonal series expansion (OSE) method, and the EOLE method.

In engineering applications, it is quite common that we have only one trajectory or just a few trajectories of the stochastic loads. With limited data, the KL-expansion-based method may not be applicable since KL expansion is based on exact correlation, mean, and variance functions. In this situation, the data-driven approach is more promising. As a data driven approach, the time series analysis has been widely used in many areas for the modeling of stochastic loading and perform prediction based on currently available data. The commonly used regression techniques for time series model include autoregressive (AR) model, moving average (MA) model, and autoregressive moving average (ARMA) model, which are suitable for stationary stochastic processes. When the stochastic process is nonstationary, the autoregressive integrated moving average (ARIMA) model is employed [40]. In this work, we mainly focus on stationary stochastic processes and use the ARMA model. An ARMA (p, q) time series model is given by

$$Y_j(t_i) = \varphi_j^{(0)} + \varphi_j^{(1)} Y_j(t_{i-1}) + \varphi_j^{(2)} Y_j(t_{i-2}) + \cdots + \varphi_j^{(p)} Y_j(t_{i-p}) + \varepsilon(t_i) - \omega_j^{(1)} \varepsilon(t_{i-1}) - \cdots - \omega_j^{(q)} \varepsilon(t_{i-q}) \tag{10}$$

in which $\varepsilon(t_i), \varepsilon(t_{i-1}), \ldots, \varepsilon(t_{i-q})$ is a sequence of independent and identically distributed random variables with zero mean and finite standard deviation σ_ε; $\varphi_j^{(0)}, \varphi_j^{(1)}, \ldots, \varphi_j^{(p)}$, and $\omega_j^{(1)}, \ldots, \omega_j^{(q)}$ are the coefficients of the time-series model $Y_j(t)$, p is the order of the AR model, and q is the order of the MA model. The random variables, $\varepsilon(t_i), \varepsilon(t_{i-1}), \ldots, \varepsilon(t_{i-q})$, can follow Weibull, normal, or other distributions. In the following discussion, unless otherwise mentioned, $\varepsilon(t_i), \varepsilon(t_{i-1}), \ldots, \varepsilon(t_{i-q})$ are assumed to follow normal distributions.

In order to predict the future realization of a stochastic process based on available data, the coefficients $\varphi_j^{(0)}, \varphi_j^{(1)}, \ldots, \varphi_j^{(p)}$, and $\omega_j^{(1)}, \ldots, \omega_j^{(q)}$ must be identified first. There are many methods available to estimate these coefficients, such as the Yule–Walker method, Burg method, covariance method, and the maximum-likelihood estimation method [40]. Next, we will discuss how the time-series model is applied in time-dependent reliability analysis.

3.3.2 Application of Time-Series Model in Time-Dependent Reliability Analysis. In order to perform time-dependent reliability analysis for problems with time-series models using the method reviewed in Sec. 2.2, the mean, standard deviation, and the autocorrelation function of time-series models need to be obtained first. The mean value of the ARMA (p, q) model is given by

$$\mu_{Y_j} = \frac{\varphi_j^{(0)}}{1 - \varphi_j^{(1)} - \cdots - \varphi_j^{(p)}} \tag{11}$$

After subtracting the mean of $Y_j(t)$ at every time instant, $Y_j(t)$ is transformed into a zero-mean time-series model. The autocovariance of the zero mean $Y_j(t)$ is computed based on the coefficients as follows [41]:

$$\gamma_k - \varphi_j^{(1)} \gamma_{k-1} - \cdots - \varphi_j^{(p)} \gamma_{k-p} = \begin{cases} (1 - \omega_j^{(1)} \psi_1 - \cdots - \omega_j^{(q)} \psi_q) \sigma_\varepsilon^2 & \text{for } k = 0 \\ -(\omega_j^{(k)} + \omega_j^{(k+1)} \psi_1 + \cdots + \omega_j^{(q)} \psi_{q-k}) \sigma_\varepsilon^2 & \text{for } k = 1, \ldots, q \\ 0 & \text{for } k \geq q+1 \end{cases} \tag{12}$$

where γ_i, $i = 0, 1, \ldots, \infty$ are the autocovariance of $Y_j(t)$ between time instant t and $t+i$; $\psi_1, \ldots, \psi_q$ are obtained from $\varphi_j^{(1)}, \ldots, \varphi_j^{(p)}$ and $\omega_j^{(1)}, \ldots, \omega_j^{(q)}$ by equating the coefficients of B^i in the following equation [41]:

$$\frac{\omega_q(B)}{\varphi_p(B)} = \frac{1 - \omega_j^{(1)} B - \omega_j^{(2)} B^2 - \omega_j^{(q)} B^q}{1 - \varphi_j^{(1)} B - \varphi_j^{(2)} B^2 - \varphi_j^{(p)} B^p} = \psi(B) = (1 + \psi_1 B + \psi_2 B^2 + \cdots) \tag{13}$$

Based on Eq. (12), the autocorrelation $\rho_{Y_j}(t, t+k)$ of $Y_j(t)$ can be obtained by dividing the autocovariance function by γ_0. In the following part, for the sake of illustration, we use $\rho_{Y_j}(k)$ to denote $\rho_{Y_j}(t, t+k)$. For $k \leq q$, the autocorrelation $\rho_{Y_j}(k)$ is estimated based on Eq. (12). For $k > q$, the autocorrelation $\rho_{Y_j}(k)$ is estimated iteratively as follows:

$$\rho_{Y_j}(k) = \varphi_j^{(1)} \rho_{Y_j}(k-1) + \varphi_j^{(2)} \rho_{Y_j}(k-2) + \cdots + \varphi_j^{(p)} \rho_{Y_j}(k-p) \tag{14}$$

With the mean (Eq. (11)), standard deviation (Eq. (12)), and autocorrelation (Eqs. (12) and (14)), the stochastic loading modeled by the ARMA model can then be applied in Sec. 2.2 for time-dependent reliability analysis.

3.3.3 Bayesian Time-Series Model. The commonly used approach to model the stochastic process based on the available data is to construct a time-series model (such as ARMA) with deterministic coefficients and a noise term. This approach does not capture the epistemic uncertainty due to limited data. By incorporating the Bayesian framework into time-series modeling, a Bayesian time-series modeling technique was developed by Ling and Mahadevan [42]. In the Bayesian time-series model, both model coefficients and noise terms are assumed to be uncertain instead of deterministic.

Assume that we have n_{ts} trajectories of a stochastic loading $Y_k(t)$ available, denote these trajectories as $\mathbf{D}_k^i$, $i = 1, 2, \ldots, n_{ts}$, where $\mathbf{D}_k^i = [Y_k^i(t_1), Y_k^i(t_2), \ldots, Y_k^i(t_{n_t})]$ and $Y_k^i(t_j)$, $j = 1, 2, \ldots, n_t$, are the ith trajectory of the stochastic loading $Y_k(t)$ at time instant t_j. For the given values of $\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k$, the standard deviation σ_ε is computed by comparing the model prediction and the observed data $\mathbf{D}_k$ as follows [42]:

$$\sigma_\varepsilon^2 = \frac{1}{n_{ts}(n_t - p - 1)} \sum_{j=1}^{n_{ts}} \sum_{i=p+1}^{n_t} [Y_k^j(t_i) - \hat{Y}_k(t_i)]^2 \tag{15}$$

where n_t is the number of observations and $\hat{Y}_k(t_i)$ is the estimation of the time-series model under given coefficients of $\boldsymbol{\varphi}_k$ and $\boldsymbol{\omega}_k$.

The likelihood, $L(\mathbf{D}_k | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)$, that we have the observed data $\mathbf{D}_k = [\mathbf{D}_k^1, \ldots, \mathbf{D}_k^{n_{ts}}]$ under the condition that the coefficients of the time-series model are $\boldsymbol{\varphi}_k$ and $\boldsymbol{\omega}_k$, which is given by

$$L(\mathbf{D}_k | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) = \prod_{i=1}^{n_{ts}} L(\mathbf{D}_k^i | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) \tag{16}$$

where

$$L(\mathbf{D}_k^i | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) = L(Y_k^i(t_1), Y_k^i(t_2), \ldots, Y_k^i(t_{n_t}) | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) \tag{17}$$

Equation (17) can be further written as

$$\begin{aligned} & L(Y_k^i(t_1), Y_k^i(t_2), \ldots, Y_k^i(t_{n_t}) | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) \\ &= L(Y_k^i(t_{n_t}) | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k, \mathbf{Y}_{-t_{nt}}) L(\mathbf{Y}_{-t_{n_t}} | \boldsymbol{\varphi}_k, \boldsymbol{\omega}_k) \\ &\approx (2\pi\sigma_\varepsilon^2)^{\frac{-(n_t - p)}{2}} \exp\left\{ - \sum_{t=p+1}^{n_t} \varepsilon_t^2 / (2\sigma_\varepsilon^2) \right\} \end{aligned} \tag{18}$$

$$\varepsilon_t = Y_k^i(t_{n_t}) - \sum_{j=1}^{p} \varphi_k^{(j)} Y_k^i(t_{n_t - j}) - \sum_{j=1}^{q} \omega_k^{(j)} \varepsilon_{n_t - j} \tag{19}$$

where $\mathbf{Y}_{-t_{n_t}} = [Y_k^i(t_1), Y_k^i(t_2), \ldots, Y_k^i(t_{t_{n_t} - 1})]$ is the stochastic loading at time instants before t_{nt}; and $e_1, e_2, \ldots, e_{n_t}$ are computed iteratively using Eq. (19).

Note that the likelihood given in Eq. (18) is usually very small. To make the computation of Eq. (18) possible, a logarithm operator can be used. Then, the joint distribution of the coefficients $\boldsymbol{\varphi}_k$ and $\boldsymbol{\omega}_k$ under given observations $\mathbf{D}_k$ is updated using Bayes' theorem as follows:

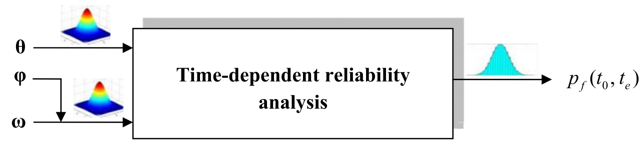


Fig. 1 UQ of time-dependent reliability analysis

$$p(\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k|\mathbf{D}_k) = \frac{L(\mathbf{D}_k|\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)\pi(\boldsymbol{\varphi}_k)\pi(\boldsymbol{\omega}_k)}{\int \cdots \int L(\mathbf{D}_k|\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)\pi(\boldsymbol{\varphi}_k)\pi(\boldsymbol{\omega}_k)\mathbf{d}\boldsymbol{\varphi}_k\mathbf{d}\boldsymbol{\omega}_k} \tag{20}$$

Similar to Eq. (9), where we obtained the posterior distributions of parameters of random variables, we also use MCMC to generate samples for $\boldsymbol{\varphi}_k$ and $\boldsymbol{\omega}_k$ based on the following proportional relationship:

$$p(\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k|\mathbf{D}_k) \propto L(\mathbf{D}_k|\boldsymbol{\varphi}_k, \boldsymbol{\omega}_k)\pi(\boldsymbol{\varphi}_k)\pi(\boldsymbol{\omega}_k) \tag{21}$$

In this paper, the slice sampling approach [43] is used to perform MCMC. When no information is available about the prior distributions of the time-series coefficients, they can be assumed to follow uniform distributions. Since random samples of the integrand in MCMC methods are correlated, in order to retain the correlation between these parameters, samples generated from MCMC will be recorded for the UQ in the following steps.

In the next section, the effects of uncertainty in random variables and stochastic loading models on the result of time-dependent reliability analysis will be investigated.

3.4 UQ of Time-Dependent Reliability Analysis.

3.4.1 Statement of Problem. As discussed in Sec. 3.1, limited data result in epistemic uncertainty in random variable parameters $\boldsymbol{\theta}$ and time-series model coefficients ($\boldsymbol{\varphi}$ and $\boldsymbol{\omega}$). These epistemic uncertainties are represented as probability distributions in the Bayesian approach, which results in two levels of uncertainty in time-dependent reliability analysis. In the outer level are the epistemic variables, i.e., the distribution parameters $\boldsymbol{\theta}$ of the random variables and the ARMA model coefficients, $\boldsymbol{\varphi}$ and $\boldsymbol{\omega}$, of the stochastic loading. The inner-level uncertainties are the aleatory uncertainties. For any given realization of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$, the time-dependent failure probability estimate $p_f(t_0, t_e)|\boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\omega}$ can be obtained by considering the aleatory variability. Since $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$ are all random, as shown in Fig. 1, the UQ is to obtain the distribution of $p_f(t_0, t_e)$ by propagating the uncertainty in $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$ through time-dependent reliability analysis.

A straightforward way is to perform time-dependent reliability analysis for each sample of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$. Since time-dependent reliability analysis needs to evaluate the limit-state function, this straightforward way is computationally prohibitive. Another possible way is to build a surrogate model for $p_f(t_0, t_e)$ as a function of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$. Figure 2 shows the general procedure for the surrogate-model-based method.

However, this surrogate model method has two main limitations: (1) the surrogate model can be constructed for the failure probability only within a specific time interval, such as $p_f(t_0, t_e)$. If we want to quantify the uncertainty in $p_f(t_0, t)$, where $t < t_e$, another surrogate model needs to be built for $p_f(t_0, t)$. (2) The dimension of the direct surrogate model is high. Assuming that there are m stochastic processes with orders of p and q (dimensions of $\boldsymbol{\varphi}$ and $\boldsymbol{\omega}$) for each process, and n random variables with nr (i.e., dimension of $\boldsymbol{\theta}$) parameters for each random variable, the dimension of the direct surrogate model, $\hat{p}_f(\boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\omega})$, will be $m \times (1 + p + q) + n \times nr$. Accurately constructing $\hat{p}_f(\boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\omega})$ is computationally very expensive, since time-dependent reliability analysis needs to be performed at each training point. In order to reduce the computational effort, in this paper, we propose an efficient approach for the UQ in time-dependent reliability analysis.

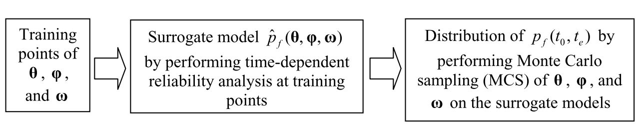


Fig. 2 General procedure of the direct surrogate-model method

3.4.2 UQ Based on Conditional Reliability Index. According to the procedure given in Sec. 2, an essential step in time-dependent reliability analysis is the search of time-instantaneous MPP under the given values of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$. Since the search of MPP for all possible realizations of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$ is very computationally expensive; in the subsequent sections, we discuss how to efficiently get the MPP under the given values of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$. Based on that, we quantify the uncertainty in time-dependent reliability analysis.

3.4.2.1 MPP search under given values of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, *and* $\boldsymbol{\omega}$ We first classify the random variables and stochastic processes into two groups as follows:

- *Group one*: Random variables that are exactly modeled (i.e., quantities with only aleatory uncertainty). We denote them as $\mathbf{X}^a$.
- *Group two*: Random variables that have uncertainty in their distribution parameters, and all stochastic processes (i. e., quantities with both aleatory and epistemic uncertainty). The modeling of the second group of variables has been discussed in Secs. 3.2 and 3.3. Here, we represent them as $\tilde{\mathbf{X}}$ and $\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t)$.

In the definition of the "group two" variables, for any given value of $\boldsymbol{\theta}$, there is a distribution of $\mathbf{X}$. Similarly, there is a stochastic process model $\mathbf{Y}(t)$ for any given values of $\boldsymbol{\varphi}$ and $\boldsymbol{\omega}$. After the classification of random variables and stochastic processes, the time-instantaneous probability of failure at t_0 becomes

$$p_f(t_0) = \Pr\{G(t_0) = g(\mathbf{X}^a, \tilde{\mathbf{X}}, \mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)) > e\} \tag{22}$$

For any given values $\tilde{\mathbf{X}} = \tilde{\mathbf{x}}$ and $\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0) = \mathbf{y}$, $p_f(t_0)|\tilde{\mathbf{x}}, \mathbf{y}$ is given by

$$p_f(t_0)|\tilde{\mathbf{x}}, \mathbf{y} = \Pr\{G(t_0) = g(\mathbf{X}^a, \tilde{\mathbf{x}}, \mathbf{y}) > e\} \tag{23}$$

The above conditional probability of failure is a time-independent problem with only aleatory variables $\mathbf{X}^a$. The MPP of $p_f(t_0)|\tilde{\mathbf{x}}, \mathbf{y}$ is obtained by solving the following optimization problem:

$$\begin{cases} \min\limits_{\mathbf{u}_{\mathbf{X}^a}} \beta^C(\tilde{\mathbf{x}}, \mathbf{y}) = \|\mathbf{u}_{\mathbf{X}^a}\| \\ g(T(\mathbf{u}_{\mathbf{X}^a}), \tilde{\mathbf{x}}, \mathbf{y}) = e \end{cases} \tag{24}$$

where $\beta^C(\tilde{\mathbf{x}}, \mathbf{y})$ is the conditional reliability index and $\Phi(-\beta^C(\tilde{\mathbf{x}}, \mathbf{y})) = p_f(t_0)|\tilde{\mathbf{x}}, \mathbf{y}$ is the conditional probability of failure.

Assume that the PDF of $\tilde{\mathbf{X}}$ is known to be $f(\mathbf{x})$ and the PDF of $\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t)$ at t_0 is $f(\mathbf{y})$, the unconditional $p_f(t_0)$ is given by

$$p_f(t_0) = \int\int (p_f(t_0)|\mathbf{x}, \mathbf{y}) f(\mathbf{x}) f(\mathbf{y}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \tag{25}$$

Since $f(\mathbf{x})$ and $f(\mathbf{y})$ vary with values of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$, directly using the above equation to obtain the MPP under any given values of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$ is still computationally intensive. To improve the efficiency, following the same principle in [44], we introduce a new random variable $U_a \sim N(0, 1)$. The new random variable has the following property:

$$\Phi(u_{p_f}) = \Pr\{U_a < u_{p_f}\} = p_f(t_0)|\mathbf{x}, \mathbf{y} \tag{26}$$

and

$$u_{p_f} = \Phi^{-1}(p_f(t_0)|\mathbf{x}, \mathbf{y}) \tag{27}$$

Substitute Eq. (27) into Eq. (25), we have

$$p_f(t_0) = \int\int\int_{U_a \le u_{p_f}} \phi(u_a) du_a f(\mathbf{x}) f(\mathbf{y}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \tag{28}$$

The above equation can be rewritten as

$$\begin{aligned} p_f(t_0) &= \Pr\{U_a \le u_{p_f}(\tilde{\mathbf{X}}, \mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0))\} \\ &= \Pr\{U_a - u_{p_f}(\tilde{\mathbf{X}}, \mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)) \le 0\} \end{aligned} \tag{29}$$

Combining Eqs. (27) and (29) yields

$$p_f(t_0) = \Pr\{U_a - \Phi^{-1}(p_f(t_0)|\tilde{\mathbf{X}}, \mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)) \le 0\} \tag{30}$$

The MPP of the above equation is obtained as follows:

$$\begin{cases} \min\limits_{\mathbf{u}} \beta(t_0) = \|\mathbf{u}\| \\ \mathbf{u} = [u_a, \mathbf{u}_{\tilde{\mathbf{X}}}, \mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}] \\ \Phi(u_a) = p_f(t_0)|T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}) \end{cases} \tag{31}$$

Since $p_f(t_0)|T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}) = \Phi(-\beta^C|T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}))$, Eq. (31) is rewritten as

$$\begin{cases} \min\limits_{\mathbf{u}} \beta(t_0) = \|\mathbf{u}\| \\ \mathbf{u} = [u_a, \mathbf{u}_{\tilde{\mathbf{X}}}, \mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}] \\ u_a = -\beta^C|T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}) \end{cases} \tag{32}$$

Equation (32) indicates that $\beta^C|T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}) = \beta^C(\tilde{\mathbf{x}}, \mathbf{y})$ needs to be evaluated during the MPP search. As shown in Eq. (24), the evaluation of $\beta^C|T(\mathbf{u}_{\mathbf{X}^{(\boldsymbol{\theta})}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)})$ will call the original limit-state function. To reduce the function evaluations of the limit-state function, we construct a surrogate model of $\hat{\beta}^C(\tilde{\mathbf{x}}, \mathbf{y})$. In this paper, the Kriging method [45,46] is used to build the surrogate model. After the surrogate model is constructed, solving Eqs. (31) and (32) does not need to call the original limit-state function anymore. The advantage of building a surrogate model for the conditional reliability index is that the surrogate model $\hat{\beta}^C(\tilde{\mathbf{x}}, \mathbf{y})$ is independent from the distributions of $\tilde{\mathbf{X}}$ and $\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)$. When the distributions of $\tilde{\mathbf{X}}$ and $\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)$ change, the surrogate model is still applicable. Assume that there are m stochastic processes with orders of p and q for each process and n random variables with nr parameters for each random variable, the dimension of the surrogate model is $m + n$.

Substituting Eq. (24) into (32), we have

$$\begin{cases} \min\ \beta(t_0) = \sqrt{\mathbf{u}^2_{\mathbf{X}^a} + \mathbf{u}^2_{\tilde{\mathbf{X}}} + \mathbf{u}^2_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}} \\ g(T(\mathbf{u}_{\mathbf{X}^a}), T(\mathbf{u}_{\tilde{\mathbf{X}}}), T(\mathbf{u}_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)})) = e \end{cases} \tag{33}$$

This implies that the $\beta(t_0)$ and $\mathbf{u}^*_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}$ obtained from Eq. (32) are the same as those obtained from the MPP search of limit-state function $G(t_0) = g(\mathbf{X}^a, \tilde{\mathbf{X}}, \mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0))$. We therefore can use them to perform the time-dependent reliability analysis under given values of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$. Note that solving Eq. (32) is based on the surrogate model of conditional reliability index (i.e., $\hat{\beta}^C(\tilde{\mathbf{x}}, \mathbf{y})$). The accuracy of the surrogate model will therefore affect the accuracy of the obtained MPP point $\mathbf{u}^*_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}$. The accuracy of the surrogate model can be improved by adding more training points. In order to guarantee the accuracy of $\mathbf{u}^*_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}$, the mean square error (MSE) of the surrogate model $\hat{\beta}^C(\tilde{\mathbf{x}}, \mathbf{y})$ needs to be checked during the construction of surrogate model.

3.4.2.2 Time-dependent reliability analysis using $\beta(t_0)$ and $\mathbf{u}^*_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}$ Before applying $\beta(t_0)$ and $\mathbf{u}^*_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}$ to the time-dependent reliability analysis, we perform the following transformation of Eq. (5):

$$\begin{aligned} \rho_L(t_0, t) &= \boldsymbol{\alpha}_{\mathbf{X}}\boldsymbol{\alpha}^T_{\mathbf{X}} + \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\rho}(t_0, t)\boldsymbol{\alpha}^T_{\mathbf{Y}} \\ &= \boldsymbol{\alpha}_{\mathbf{X}}\boldsymbol{\alpha}^T_{\mathbf{X}} + \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\alpha}^T_{\mathbf{Y}} + \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\rho}(t_0, t)\boldsymbol{\alpha}^T_{\mathbf{Y}} - \boldsymbol{\alpha}_{\mathbf{Y}}\boldsymbol{\alpha}^T_{\mathbf{Y}} \\ &= 1 + \frac{1}{\beta^2}(\mathbf{u}^*_{\mathbf{Y}}\boldsymbol{\rho}(t_0, t)\mathbf{u}^{*\mathrm{T}}_{\mathbf{Y}} - \mathbf{u}^*_{\mathbf{Y}}\mathbf{u}^{*\mathrm{T}}_{\mathbf{Y}}) \end{aligned} \tag{34}$$

where the elements of $\boldsymbol{\rho}(t_0, t)$ are given in Eq. (6), which are obtained based on the correlation analysis of time-series models under given values of $\boldsymbol{\varphi}$ and $\boldsymbol{\omega}$.

With $\beta(t_0)$, $\mathbf{u}^*_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}$, and Eq. (34), the correlation matrix given in Eq. (7) is obtained. Using those results, the time-dependent probability of failure is estimated using the method presented in Sec. 2. For each sample of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\omega}$ generated from MCMC, based on the surrogate model of β^C, the corresponding $\beta(t_0)$ and $\mathbf{u}^*_{\mathbf{Y}^{(\boldsymbol{\varphi},\boldsymbol{\omega})}(t_0)}$ are obtained from Eq. (32). The associated time-dependent probability of failure is then computed, and the uncertainty in the time-dependent reliability analysis is quantified.

3.5 Error Analysis. There are basically four approximations implemented in the proposed framework for UQ in time-dependent reliability analysis: (1) linearization of the limit state function in FORM, (2) the sampling approach used to estimate the probability of failure based on stochastic expansion, (3) the surrogate model of conditional reliability index, and (4) the fast integration method.

The error in the first approximation is problem-dependent, affected by the nonlinearity of the response function at the MPP. It can be quantified only by comparing FORM with the MCS result. The proposed method is therefore mainly for problems in which FORM is accurate for time-instantaneous reliability analysis. The error in the second approximation comes from two sources, namely, expansion of stochastic process and sampling statistical uncertainty. The error due to the expansion of stochastic process is negligible, since a large number of expansion terms are used. The statistical uncertainty is quantified by

$$\mathrm{COV}_{p_f} = \sqrt{(1 - \hat{p}_f)/\hat{p}_f/N} \tag{35}$$

where $\hat{p}_f$ is the failure probability estimate obtained from the sampling-based method and N is the number of samples used. Equation (35) shows that in order to reduce the error introduced by the statistical uncertainty, N therefore needs to be chosen as a large number.

The error of the third approximation comes from the prediction uncertainty of the surrogate model. As discussed in the last section, the accuracy of the surrogate model needs to be checked to reduce the effects of surrogate-model uncertainty on the final time-dependent reliability estimates. In terms of the fourth approximation (i.e., fast integration), it has been shown in Eqs. (31)–(33) that the MPP obtained from the fast integration is the same as that obtained from the original optimization model. The fast integration does not introduce extra error into the analysis.

Based on the above error analysis, it is concluded that the value of N should be large, and the MSE of the conditional reliability index surrogate model needs to be checked to guarantee the accuracy of the UQ in time-dependent reliability analysis.

3.6 Numerical Procedure. The overall procedure of UQ in time-dependent reliability analysis due to limited data is shown in Fig. 3 and summarized as follows:

- *Module one*: UQ of $\tilde{\mathbf{X}}$ and $\mathbf{Y}(t)$ using the Bayesian approach. Posterior distributions and samples of $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\omega}$ are obtained from MCMC sampling.
- *Module two*: Construction of the surrogate model $\hat{\beta}^C(\tilde{\mathbf{x}}, \mathbf{y})$. Time-independent reliability analyses are performed at specific training points of $\tilde{\mathbf{x}}$ and $\mathbf{y}$ using Eq. (24). The training

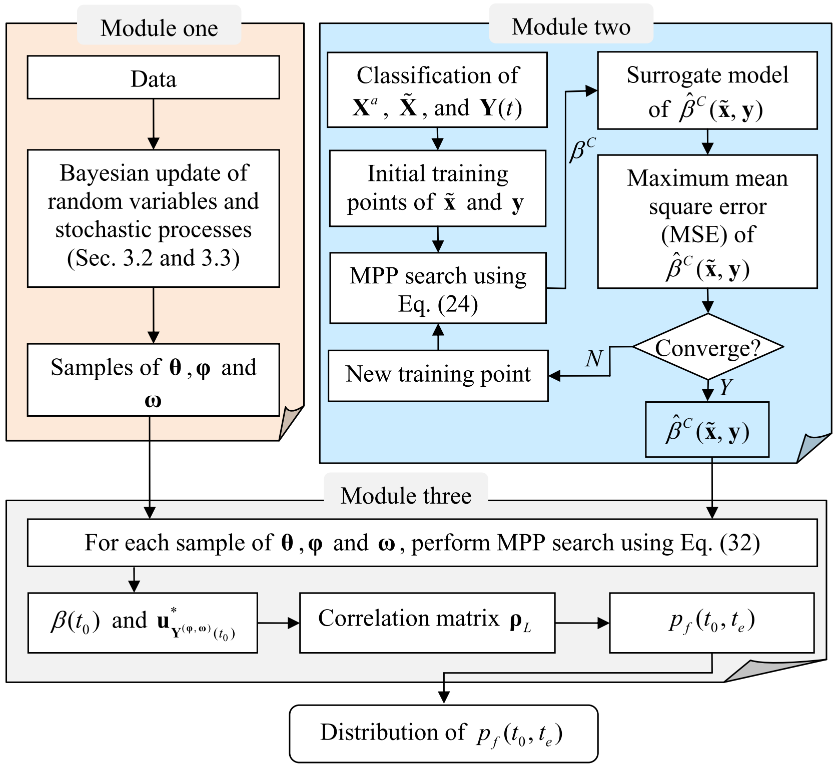


Fig. 3 Overall framework of UQ in time-dependent reliability analysis

points are generated using the Hammersley sampling approach [47]. Training points are progressively added until the convergence criterion of MSE for $\hat{\beta}^C(\tilde{\mathbf{x}}, \mathbf{y})$ is satisfied.

- *Module three*: UQ of $p_f(t_0, t_e)$. For samples generated from module one, $\beta(t_0)$ and $\mathbf{u}^*_{\mathbf{Y}(\boldsymbol{\varphi},\boldsymbol{\omega})(t_0)}$ are obtained using Eq. (32) based on the surrogate model built in module two. $p_f(t_0, t_e)$ is then approximated using the method presented in Sec. 3.

4 Numerical Examples

In this section, two examples, which include a mathematical example and an engineering application example, are used to demonstrate the proposed UQ framework. Each example is solved using three methods given as follows:

- "*True*" p_f: The probability of failure obtained from Monte Carlo simulation (MCS), which is performed based on the assumed "true" random variable distributions and time-series models of stochastic loadings.
- *Only aleatory*: Random variables and time-series models are reconstructed from observations. Based on the constructed deterministic time-series models and random variables, the time-dependent probability of failure is estimated without considering epistemic uncertainty.
- Aleatory + epistemic: The proposed UQ framework for time-dependent reliability analysis is used to consider the effect of limited data.

4.1 Mathematical Example.

Consider the function

$$g(t) = X_1 + X_2 - Y_1(t) \tag{36}$$

where $X_1 \sim N(70, 10^2)$ and $X_2 \sim N(65, 5^2)$ are random variables and $Y_1(t)$ is a stochastic process given by

$$\begin{aligned} Y_1(t_i) = \varphi^{(0)} + \varphi^{(1)} Y(t_{i-1}) + \varphi^{(2)} Y(t_{i-2}) + \varphi^{(3)} Y(t_{i-3}) \\ + \varepsilon(t_i) + \omega^{(1)} \varepsilon(t_{i-1}) + \omega^{(2)} \varepsilon(t_{i-2}) \end{aligned} \tag{37}$$

where $\varphi^{(0)} = 60$; $\varphi^{(1)} = 0.7231$; $\varphi^{(2)} = -0.1256$; $\varphi^{(3)} = 0.0262$; $\omega^{(1)} = 0.3$; $\omega^{(2)} = 0.12$; and $\varepsilon \sim N(0, 10^2)$.

The following time-dependent probability of failure needs to be evaluated:

$$p_f(t_0, t_e) = \Pr\{g(\tau) = X_1 + X_2 - Y_1(\tau) > 0, \exists \tau \in [t_0, t_e]\} \tag{38}$$

where $t_0 = 0$ and $t_e = 30$.

Suppose that we do not know the exact models of $Y_1(t)$ and X_2. Instead, they are reconstructed based on available experimental data as shown in Fig. 4. One hundred cycles of $Y_1(t)$ and 100 samples of X_2 are assumed to be collected and plotted, based on which $Y_1(t)$ and X_2 are reconstructed. When the data of $Y_1(t)$ and X_2 are collected, to account for noise in the sensors and variability in the experimental conditions, noise terms $\varepsilon_Y \sim N(0, 1^2)$ and $\varepsilon_X \sim N(0, 0.5^2)$ are added to the data of $Y_1(t)$ and X_2, respectively.

We then perform time-dependent reliability analysis using MCS; the method considers only aleatory uncertainty, and the proposed method. In the proposed method, $N = 2 \times 10^6$. It means that the COV_{p_f} is less than 0.005, since the probability of failure is close to 0.1. We also checked the MSE of the conditional reliability index surrogate model as shown in Fig. 5. It shows that the uncertainty in the surrogate model prediction is negligible with six training points. Figure 6(*a*) presents the updated posterior distributions of $p_f(t_0, t_e)$ up to 30 cycles obtained from the proposed method. Figure 6(*b*) shows the comparison of $p_f(0, 30)$ obtained from the three methods. The results show that the proposed method is able to effectively quantify the uncertainty in $p_f(t_0, t_e)$. With limited experimental data, there exists significant uncertainty in the result of time-dependent reliability analysis.

In order to investigate how the number of experimental data affects the uncertainty in the time-dependent reliability prediction as well as the ability of the proposed method to update the results of time-dependent reliability analysis, we increase the number of cycles of stochastic load history data from 100 to 200 and 500. We then update the time-dependent probability of failure distribution using the proposed UQ framework. Figure 7 shows the comparison of posterior distributions of $p_f(0, 30)$.

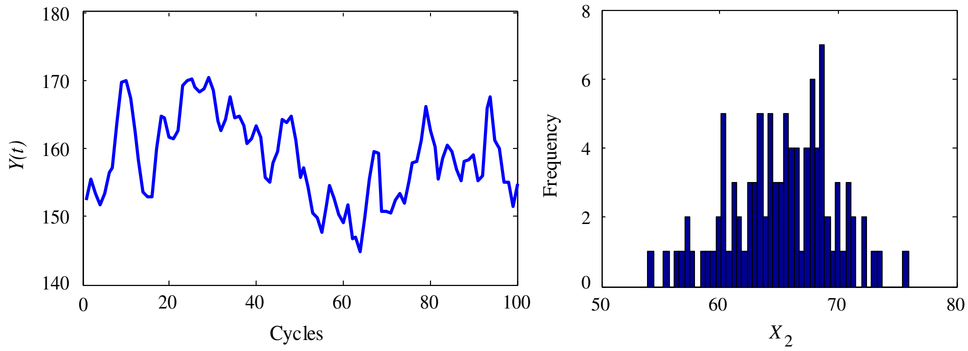


Fig. 4 Experimental data of $Y_1(t)$ and X_2

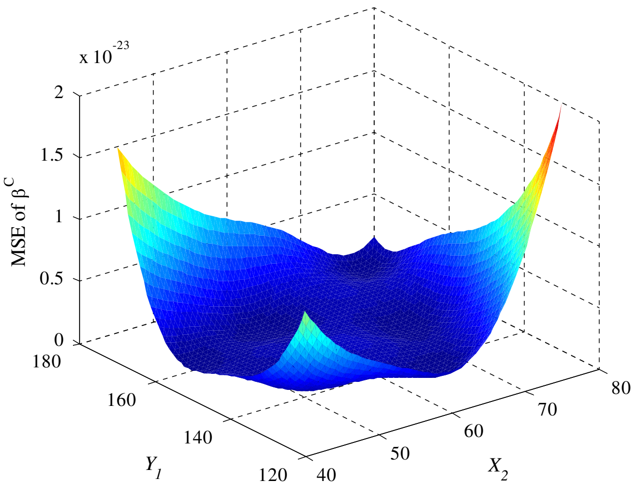


Fig. 5 MSE of the conditional reliability index

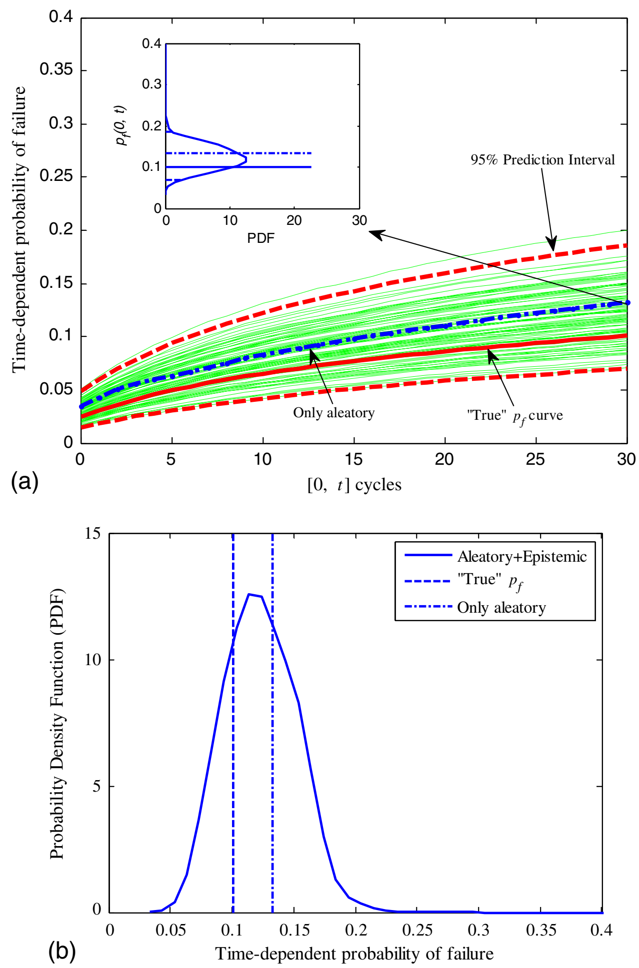


Fig. 6 Results based on 100 cycles of $Y_1(t)$ and 100 samples of X_2. (a) Trajectories of $p_f(t_0, t_e)$ up to 30 cycles (inner figure is the distribution at $t = 30$). (b) $p_f(0,30)$ obtained from three methods

The results show that the uncertainty of time-dependent reliability analysis is reduced when new observations are collected. The effectiveness of the proposed method in quantifying the uncertainty in reliability prediction is thus demonstrated. Table 1 presents the number of function (NOF) evaluations of the original limit-state function for the case of 100 cycles of stochastic load history and 100 samples of random variables. It shows that the NOF evaluations of the proposed method is slightly larger than the only aleatory method (which considers only aleatory uncertainty). This phenomenon is partly due to the linear property of the problem, since the surrogate modeling is easier for the linear problem. Considering that the proposed UQ method is able to account for both epistemic and aleatory uncertainty, the proposed method is still very efficient. In the reliability analysis part, the MPP is obtained using the fmincon in matlab to solve the optimization model given in Eq. (24).

4.2 Beam Subjected to Stochastic Loading History.

A beam subjected to a stochastic loading history $F(t)$ is shown in Fig. 8. This example is modified from [11].

The limit-state function of the beam example is given by

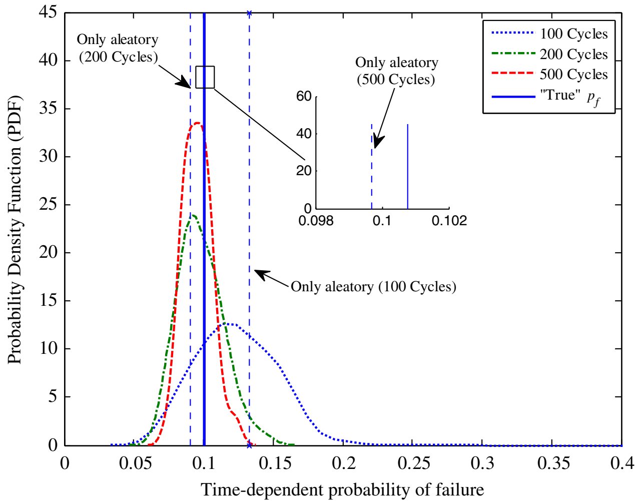
$$g(\mathbf{X}, \mathbf{Y}(t)) = \left(\frac{F(t)L_b}{4} + \frac{\rho_{st} a_0 b_0 L_b^2}{8}\right) - \frac{1}{4} a_0 b_0^2 \sigma_u \tag{39}$$

where σ_u is the ultimate strength; ρ_{st} is the density; and L is the length of the beam. Table 2 gives the parameters and random variables of this example.

The historical data of $F(t)$ over the past 200 cycles are assumed to be available. We want to predict the reliability of the beam in the future 50 cycles. The time-dependent probability of failure in the future 50 cycles is given by

$$p_f(t_0, t_e) = \Pr\{g(\mathbf{X}, \mathbf{Y}(\tau)) > 0, \exists \tau \in [t_0, t_e]\} \tag{40}$$

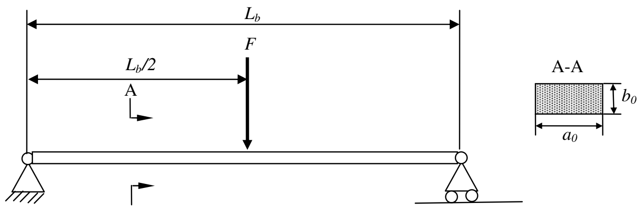
in which $t_0 = 0$ and $t_e = 50$.



F7:1 **Fig. 7 Comparison of posterior distributions of $p_f(0{,}30)$**

Table 1 Comparison of the NOF evaluations

	Method	Only aleatory	Aleatory + epistemic
T1:1	Method	Only aleatory	Aleatory + epistemic
T1:2	NOF	70	106



F8:1 **Fig. 8 Beam subjected to a concentrated stochastic load**

Table 2 Variables and parameters of the beam example

	Variable	Mean	Standard deviation	Distribution
T2:1	Variable	Mean	Standard deviation	Distribution
T2:2	a_0	0.2 m	0.005 m	Normal
T2:3	b_0	0.042 m	2×10^{-3} m	Normal
T2:4	σ_u	2.4×10^8 Pa	1.1×10^7 Pa	Normal
T2:5	$F(t)$	Stochastic loading constructed from historical data		
T2:6	L_b	6 m	0	Deterministic
T2:7	ρ_{st}	78.5 kN/m^3	0	Deterministic

Assume that the historical data of $F(t)$ in the past 200 cycles are generated from an underlying time-series model $Y(t)$ and $F(t) = 43Y_1(t)$. $Y_1(t)$ is then given by

$$Y_1(t_i) = \varphi^{(0)} + \varphi^{(1)}Y(t_{i-1}) + \varphi^{(2)}Y(t_{i-2}) + \varphi^{(3)}Y(t_{i-3}) + \varepsilon(t_i) + \omega^{(1)}\varepsilon(t_{i-1}) + \omega^{(2)}\varepsilon(t_{i-2}) \quad (41)$$

in which $\varphi^{(0)} = 70$; $\varphi^{(1)} = 0.7315$; $\varphi^{(2)} = -0.1421$; $\varphi^{(3)} = 0.0612$; $\omega^{(1)} = 0.34$; $\omega^{(2)} = 0.13$; and $\varepsilon \sim N(0, 12^2)$.

Similarly, we assume that the parameters of σ_u are unknown and they need to be estimated from the experimental data. Assume that 200 samples of σ_u are collected. The noise terms from sensor and experimental variability for $Y(t)$ and σ_u are $\varepsilon_Y \sim N(0, 1.2^2)$ and $\varepsilon_\sigma \sim N(0, (1.1 \times 10^6)^2)$. Figure 9 shows the historical and experimental data of $F(t)$ and σ_u.

We then perform time-dependent reliability analysis for the beam based on the available data. Similar to Example 1, we checked the COV_{p_f} and the MSE (as shown in Fig. 10) of the conditional reliability-index surrogate model. The COV_{p_f} is also less than 0.005, and the MSE of the surrogate model prediction is negligible (less than 0.2% of the mean prediction) with six training points.

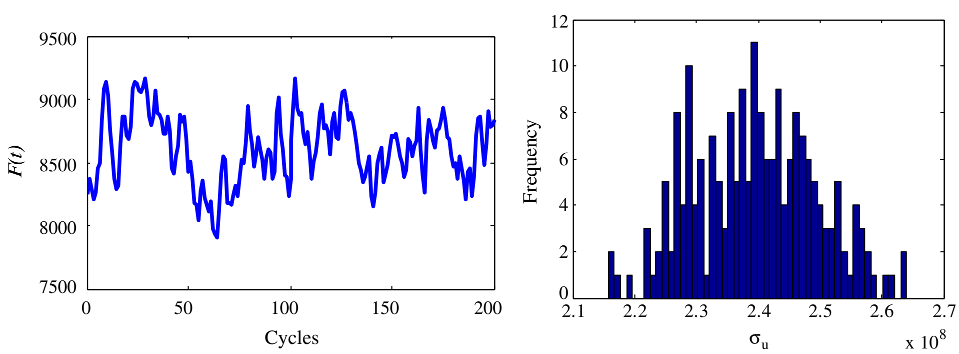
Figure 11(*a*) shows the updated posterior distributions of $p_f(t_0, t_e)$ obtained from the proposed method up to 50 cycles. Figure 11(*b*) presents the comparison of $p_f(0, 50)$ obtained from MCS, the proposed method, and the only aleatory method. Table 3 gives the NOF evaluations of the original limit-state function for the case of 200 cycles of stochastic load and 200 samples of random variables. It requires around 69 function evaluations per training point in average.

The results indicate that there is a large uncertainty in the prediction of time-dependent probability of failure with limited data on the stochastic loading. The NOF evaluations of the proposed method is about twice that of the method, which considers only the aleatory uncertainty. Since the proposed method accounts for both epistemic and aleatory uncertainties while the aleatory method considers only the aleatory uncertainty, the proposed method is still very efficient. To investigate the effects of the number of experimental data, similar to Example 1, we increase the numbers of cycles of collected data from 200 to 1000 and 2000 for $F(t)$.

With more observations collected, we update the posterior distributions of $p_f(t_0, t_e)$ using the proposed method. Figure 12 shows the updated posterior distributions of $p_f(0, 50)$.

The results imply that the uncertainty of time-dependent probability of failure prediction has been reduced slightly with more observations of the stochastic load history. The improvement, however, is not significant. We then collect more experimental data for the ultimate strength σ_u. The number of observations of σ_u is increased from 200 to 2000. The number of observations of $F(t)$ is still 2000. Figure 13 shows the updated posterior distribution of time-dependent probability of failure.

Figure 13 shows that the uncertainty in the reliability prediction is reduced with more observations. Other conclusions similar to the



F9:1 **Fig. 9 Historical and experimental data of $F(t)$ and σ_u**

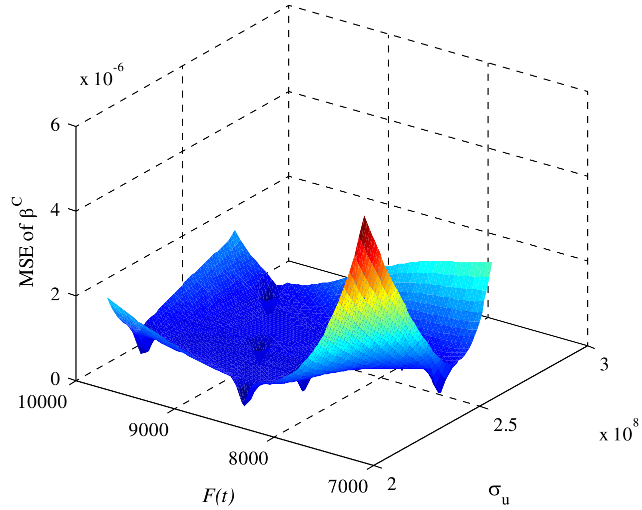


Fig. 10 MSE of the conditional reliability index

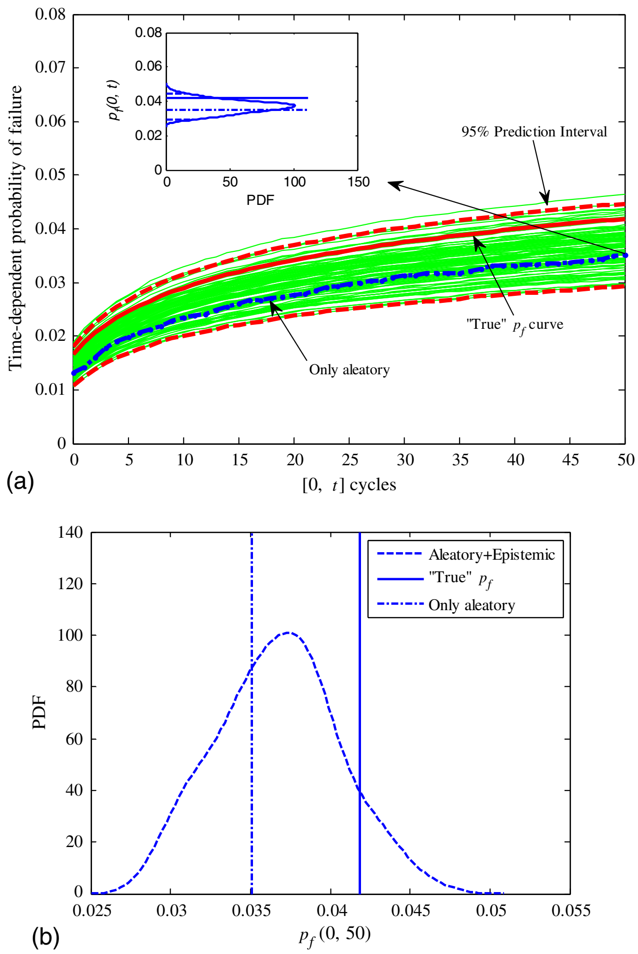


Fig. 11 Results of $p_f(t_0,t_e)$ obtained based on 200 cycles of $F(t)$ and 200 samples of σ_u. (a) Updated posterior distributions of $p_f(t_0,t_e)$ up to 50 cycles. (b) Comparison of $p_f(0,50)$ obtained from three methods

Table 3 Comparison of the NOF evaluations

Method	Only aleatory	Aleatory + Epistemic
NOF	182	412

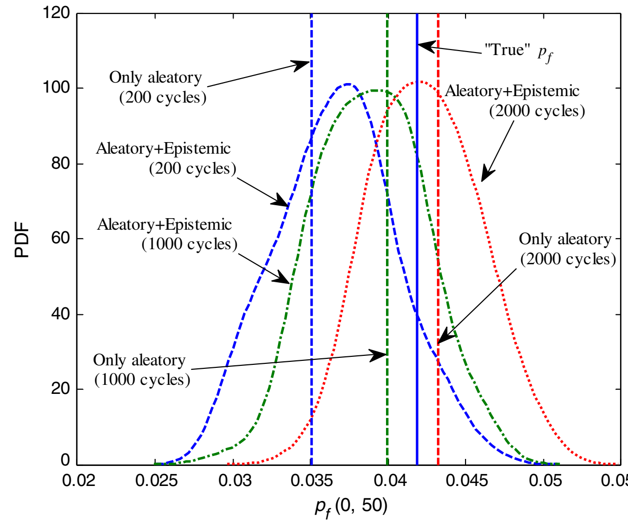


Fig. 12 Comparison of updated posterior distributions of $p_f(0,50)$

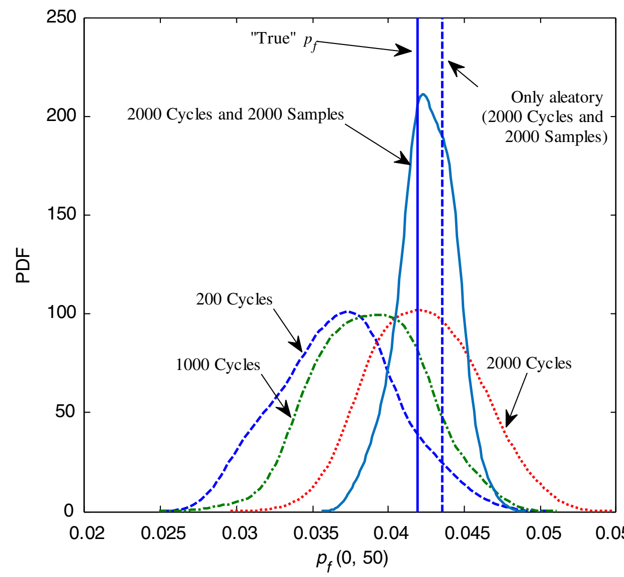


Fig. 13 Updated posterior distributions with more observations of σ_u

first example can also be drawn. Thus, the proposed method can quantify the uncertainty in time-dependent reliability analysis effectively.

5 Conclusion

Time-dependent reliability gives the degradation of reliability over time. It is directly related to safety inspection, maintenance scheduling, and lifecycle-cost optimization. The stochastic loads and random variables are assumed to be exactly known in the traditional analysis methods, i.e., only aleatory uncertainty (natural variability) is considered. In practical applications, it is common that the collected data or observations are too limited to accurately model the stochastic loads and random variables. Accounting for both uncertainties due to limited data and natural variability is a challenging and meaningful issue in time-dependent reliability analysis.

A UQ framework is proposed in this paper for time-dependent reliability analysis by incorporating both epistemic uncertainty (due to limited data) and aleatory uncertainty. The random variable distributions and stochastic loading history models are constructed based on collected observations. The Bayesian approach is used to quantify the epistemic uncertainty in the modeling of stochastic loading and random variables due to limited data. Through the construction of a surrogate model for the time-instantaneous conditional reliability index, the effects of epistemic uncertainty due to limited data on the time-dependent reliability analysis are efficiently quantified. A mathematical example and an engineering application example demonstrated the effectiveness of the proposed method.

Since the time-dependent probability of failure is presented as a probability distribution in the proposed method, how to guide decision such as design optimization and inspection scheduling using the obtained probability distribution is one of our future investigations. The proposed method currently focuses only on problems with stationary stochastic process loadings. In the future, we will also investigate Bayesian time-dependent reliability analysis method for problems with nonstationary loading history. Application of the proposed method to more complicated and sophisticated engineering systems needs to be studied as well in the future.

Acknowledgement

The research reported in this paper was supported by the Air Force Office of Scientific Research (Grant No. FA9550-15-1-0018, Technical Monitor: Dr. David Stargel). The support is gratefully acknowledged.

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